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Combinatorial Reciprocity Theorems

Homework # 6 — due June 2, **before** the lecture

Exercise 1. For $n \in \mathbb{N}$ fixed consider the map $f_{\leq} : \mathcal{B}_k \to \mathbb{N}$ with

$$f_{\leq}(T) \ = \ |T|^n.$$

Show that

$$k!\, S(n,k) \ = \ (f_{\leq}\, \mu_{\mathcal{B}_k})([k])$$

where $S(n,k)$ is the Stirling number of the second kind.
[Hint: $k!S(n,k)$ counts the number of surjective maps $[n] \to [k]$.]

(4 points)

Exercise 2. i) Let P be a graded poset with $\hat{0}$ and $\hat{1}$. The **rank** of an element $x \in P$ is defined by

$$\text{rk}(x) := l(\hat{0}, x).$$

Show that P is Eulerian if and only if the number of elements of odd rank equals the number of elements of even rank in every interval.

ii) Suppose P and Q are two Eulerian posets and let $P' = P \setminus \{\hat{0}\}$ and $Q' = Q \setminus \{\hat{0}\}$. Show that $(P' \times Q') \cup \{\hat{0}\}$ is Eulerian.

(3+4 points)

Exercise 3. Let $P \subset \mathbb{R}^d$ be a polytope and $C = \hom(P) = \{(x,t) : x \in tP, t \geq 0\}$ its homogenization. Show that for every $i \geq -1$, there is a bijection between the i-faces of P and the $(i+1)$-faces of C.

(4 points)