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Combinatorial Reciprocity Theorems

Homework # 4 — due May 19, **before** the lecture

Exercise 1. Let P, Q be two finite posets. Then $P \times Q$ is a poset by defining

$$(p_1, q_1) \ \preceq_{P\times Q} \ (p_2, q_2)$$

if and only if $p_1 \preceq_P p_2$ and $q_1 \preceq_Q q_2$ for $p_1, p_2 \in P$ and $q_1, q_2 \in Q$.

i) Show that every interval of $P \times Q$ is of the form $[p, p']_P \times [q, q']_Q$.

ii) Show that we have

$$\mu_{P\times Q}((p_1, q_1), (p_2, q_2)) \ = \ \mu_P(p_1, p_2) \cdot \mu_Q(q_1, q_2)$$

for all $p_1, p_2 \in P$ and $q_1, q_2 \in Q$.

iii) By using ii) show that the Möbius function of the Boolean lattice $\mathcal{B}_k = (2^{[k]}, \subseteq)$ satisfies

$$\mu_{\mathcal{B}_k}(S, T) \ = \ (-1)^{|T\setminus S|}$$

for $S \subseteq T \in \mathcal{B}_k$.

iv) Bonus: Let $G = (E, V)$ be a graph. Consider the poset $\mathcal{S} \subseteq 2^E$ of all cycle free subgraphs ordered by inclusion. Compute the Möbius function of $\mathcal{S}$.

(1+2+2+2 points)

Exercise 2. Let Π be a finite poset. Recall that $\zeta = \delta + \eta$ where $\eta(x, y) = 1$ if and only if $x \prec y$.

i) Show that for $x \preceq y$

$$\eta^n(x, y) = \#\{x = x_0 \prec x_1 \prec \cdots \prec x_n = y\},$$

the number of *strict* chains of length n in the interval $[x, y]$.

ii) Infer that η is nilpotent, that is, $\eta^n \equiv 0$ for all $n > n_0$. What is the smallest n_0?

iii) Bonus: Can you give an interpretation for $(2\delta - \zeta)^{-1}(x, y)$?

iv) Show that $\eta^r_{\mathcal{J}(\Pi)}(\emptyset, \Pi)$ equals the number of surjective order preserving maps $\Pi \to [r]$.

(2+2+2+2 points)

Exercise 3. Prove **Birkhoff's representation theorem**:

Every finite distributive lattice is isomorphic to the poset of order ideals of some finite poset.

The definition of "distributive lattice" is in the book draft.

(Hint: Given a distributive lattice Π, consider the subposet Θ consisting of all **join-irreducible** elements, i.e. those elements $x \in \Pi$, for which $x \neq y \vee z$ for all $y \neq z$. Prove, that Π is isomorphic to $\mathcal{J}(\Theta)$.)

(4 points)