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SS2014

Combinatorial Reciprocity Theorems

Homework # 3 — due May 12, **before** the lecture

Exercise 1. Let $P, Q_1, \dots, Q_m \subset \mathbb{R}^2$ be lattice polygons, such that Q_i lies in the interior of P for all $1 \leq i \leq m$ and $Q_i \cap Q_j = \emptyset$ for all $i \neq j$. Let $\mathcal{S} = P \setminus \bigcup_{i=1}^m Q_i^\circ$ be a "polygon with m holes".

i) Show that the Ehrhart function $\mathrm{ehr}_{\mathcal{S}}(n)$ of $\mathcal{S}$ agrees with a polynomial for $n \geq 1$.

ii) What are the coefficients of the Ehrhart polynomial $\mathrm{ehr}_{\mathcal{S}}(n)$?

iii) What does $\mathrm{ehr}_{\mathcal{S}}(-n)$ count?

(1+3+2 points)

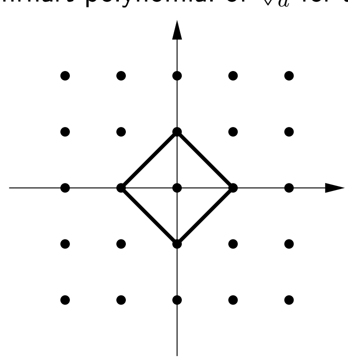
Exercise 2. The d-dimensional **crosspolytope** is the convex polytope

$$\Diamond_d = \left\{x \in \mathbb{R}^d : |x_1| + |x_2| + \cdots + |x_d| \ \leq \ 1\right\}$$

i) Compute the Ehrhart polynomial of $\Diamond_2$.

ii) Compute the Ehrhart polynomial of the octahedron $\Diamond_3$.

iii) Bonus: Compute the Ehrhart polynomial of $\Diamond_d$ for all $d \geq 1$.



(2+3+4 points)

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Exercise 3. A **linear extension** of a finite poset Π is a bijective strictly order preserving map $\lambda\colon \Pi \to [d]$ where $d = |\Pi|$.

i) Show that every finite poset Π has a linear extension.

ii) Show that linear extensions are in bijection with strict chains of order ideals of length $|\Pi|$.

(2+2 points)