Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal
Katharina Jochemko

SS2014

Combinatorial Reciprocity Theorems

Homework # 2 — due May 5, **before** the lecture

Exercise 1. i) Let $\mathcal{S} = \text{conv}\{(a_1, b_1), (a_2, b_2)\}$ with $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, be a **lattice segment**. Show that

$$\text{ehr}_{\mathcal{S}}(n) = Ln + 1$$

where $L = \gcd(|a_2 - a_1|, |b_2 - b_1|)$ is the **lattice length** of $\mathcal{S}$. Conclude further that $-\text{ehr}_{\mathcal{S}}(-n)$ is the number of lattice points in $n\mathcal{S}$ other than the endpoints, i.e.

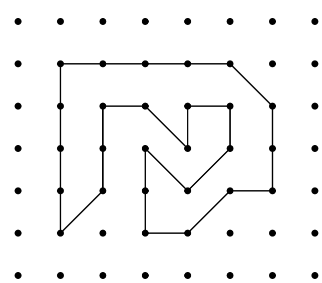
$$(-1)^{\dim(\mathcal{S})}\text{ehr}_{\mathcal{S}}(-n) = \text{ehr}_{\mathcal{S}^\circ}(n).$$

What happens if we allow $\mathcal{S}$ to have endpoints with rational coordinates?

ii) Let $\mathcal{O}$ be a closed polygonal lattice path, i.e. the union of lattice segments, such that any two segments intersect in at most one lattice point, and that topologically $\mathcal{O}$ is a closed curve. Show that

$$\text{ehr}_{\mathcal{O}}(n) = Ln$$

where L is the number of lattice points on $\mathcal{O}$.



Bonus: What happens, if we allow overlaps?

(3+3+1 points)

(continued on backside)

Exercise 2. Let Π be a finite poset. A **chain** of length r is a collection $p_0, \ldots, p_r \in \Pi$ such that

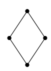
$$p_0 \prec p_1 \prec \cdots \prec p_r.$$

Show that $\Omega^\circ_\Pi(k) = 0$ if and only if Π has a chain of length $> k$.

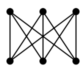
(3 points)

Exercise 3. Determine the strict order polynomial $\Omega^\circ_\Pi(n)$ if Π is a poset with Hasse diagram

i) the "diamond".



ii) the complete bipartite graph $\mathcal{K}_{3,3}$.



iii) Bonus: the complete bipartite graph $\mathcal{K}_{l,m}$ for $l, m \geq 1$.

iv) Enumerate all order polynomials of degree 3.

(2+2+2+2 points)