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Combinatorial Reciprocity Theorems

Homework # 10 — due July 14, **before** the lecture

Exercise 1. Let $C_n = [0,1]^n$ be the unit cube. For a permutation σ of $[n]$ let

$$F_\sigma \ := \ \{\phi \in \mathbb{R}^n : 0 \ \leq \ \phi_{\sigma(1)} \ \leq \ \phi_{\sigma(2)} \ \leq \ \cdots \ \leq \ \phi_{\sigma(n-1)} \ \leq \ \phi_{\sigma(n)} \ \leq \ 1\} \quad \subset \ \mathbb{R}^n.$$

i) Determine the vertices of F_σ.

ii) Let Σ be the collection of all F_σ and their faces. Show that Σ is the a triangulation of C_n.

iii) Give a description for the lower dimensional polytopes in Σ.

(2+4+2 points)

Exercise 2. Bonus: In this exercise you will show that the triangulation Σ of C_n is *regular*. That is, that there are heights $\omega(v)$ for $v \in \{0,1\}^n$ such that Δ is the complex of bounded faces of

$$C_n^\omega \ = \ \mathrm{conv}\{(v, \omega(v)) \ : \ v \in \{0,1\}^n\} \ + \ \mathbb{R}_{\geq 0}\, e_{n+1}.$$

For a vertex $v \in \{0,1\}^n$ of the cube, define $\omega(v) = |v|(n - |v|)$ where $|v| = \sum_i v_i$. For σ find the unique hyperplane $H_\sigma = \{c^T x + c_{n+1}x_{n+1} \ = \ \delta\}$ such that

$$c^T v \ + \ c_{n+1}\omega(v) \ \geq \ \delta$$

for all $v \in \{0,1\}^n$ and with equality if and only if $v \in F_\sigma$.
[Hint: Consider first the case $\sigma = 123\ldots d$ and then use the symmetries of the cube.]

(5 points)

Exercise 3. Let $G = (V, E)$ be a finite directed graph and define $A \in \{0,-1,1\}^{V \times E}$ by

$$A_{ve} \ = \ \begin{cases} 1, & \text{if } e = uv, \\ -1, & \text{if } e = vu, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

i) For $I \subseteq E$ denote by G_I the sub-graph spanned by the collection of edges and by A_I the sub-matrix with columns indexed by I. Show that A_I has full column rank $|I|$ if and only if G_I has no cycle.

[Hint: For a cycle, construct and element in the kernel of A_I.]

ii) If G_I is cycle free and b is an integral vector, show that if $A_I\, f = b$ has a solution, then f is integral.

[Hint: If v is a vertex of G_I with only one incident edge $e \in I$, then f_e is $\pm b_v$.]

iii) For $b \in \mathbb{Z}^V$ show that

$$P_G(b) \;=\; \{f \in [0,1]^E : A\, f = b\}$$

is either empty or a lattice polytope. If $p \in P_G(b)$ is a vertex, consider $I = \{i : 0 < p_i < 1\}$.

(3+2+2 points)