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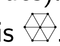
SS2014

Combinatorial Reciprocity Theorems

Homework # 1 — due April 28, **before** the lecture

Exercise 1. Find the chromatic polynomials of

i) the wheel with k spokes (and $k+1$ nodes).
For example, the wheel with 6 spokes is .

ii) the cycle on k nodes.

iii) the path on k nodes.

(3+2+1 points)

Exercise 2. Find two simple non-isomorphic graphs G and H with $\chi_G(n) = \chi_H(n)$.
G and H are isomorphic if there is a bijection $\phi : V(G) \to V(H)$ such that

$$uv \in E(G) \quad \Longleftrightarrow \quad \phi(u)\phi(v) \in E(H).$$

Bonus: Can you find many (polynomial, exponential) such examples in the number of nodes?
(Bonus)^2: Can you make your examples arbitrarily high connected?

(3+2+2 points)

Exercise 3. The complete bipartite graph $K_{r,s}$ is the graph with vertex set $V = \{1, \dots, r, 1', \dots, s'\}$ and edge set $E = \{ij' \colon i \in [r], j \in [s]\}$.

i) Determine $\chi_{K_{3,3}}(n)$.

ii) Determine $\chi_{K_{r,s}}(n)$ for all $r, s \geq 1$.

(*Hint: Proper n-colorings of $K_{r,s}$ correspond to pairs (f, g) of maps $f \colon [r] \to [n]$ and $g \colon [s] \to [n]$ with disjoint ranges.*)

(3+4 points)

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Exercise 4. Prove that the complete graph K_d (the graph on d nodes with all possible edges between them) has exactly $d!$ acyclic orientations.

(2 points)

Exercise 5. Let $G = (V, E)$ be an oriented graph and $n \geq 2$.

i) Let $f : E \to \mathbb{Z}_n$ be a nowhere-zero $\mathbb{Z}_n$-flow and let $e \in E$. Show that

$$f : E \setminus \{e\} \to \mathbb{Z}_n$$

is a nowhere-zero $\mathbb{Z}_n$-flow on the contraction G/e.

ii) For $S \subseteq V$ let $E^{\mathsf{in}}(S)$ be the *in-coming* edges, i.e., $u \to v$ with $v \in S$ and $u \in V \setminus S$ and $E^{\mathsf{out}}(S)$ the *out-going* edges. Show that $f : E \to \mathbb{Z}_n$ is a nowhere-zero $\mathbb{Z}_n$-flow if and only if

$$\sum_{e \in E^{\mathsf{in}}(S)} f(e) \;=\; \sum_{e \in E^{\mathsf{out}}(S)} f(e)$$

for all $S \subseteq V$.

iii) Infer that $\phi_G \equiv 0$ if G has a bridge.

(1+3+1 points)

Exercise 6. Let $G = (E, V)$ be a bridgeless graph. Prove that

i) $\phi_G(n)$, the number of nowhere-zero $\mathbb{Z}_n$-flows, is a polynomial in n.

ii) the degree of $\phi_G(n)$ is $|E| - |V| + 1$.

iii) Bonus: $\phi_G(n) \leq \phi_G(n+1)$ for all $n \in \mathbb{Z}_{>0}$.

(3+2+1 points)

We are happy to answer your upcoming questions at any time and give additional hints where necessary.