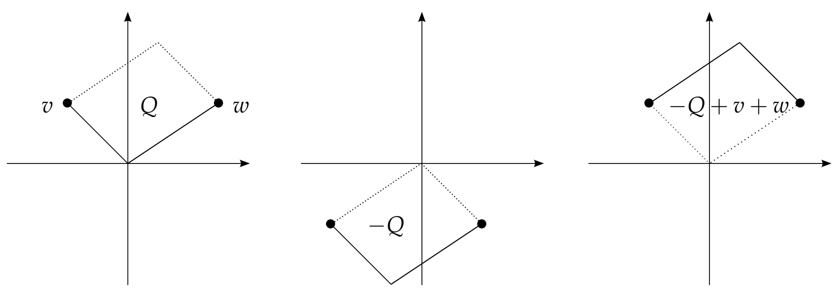
Matthias Beck • Raman Sanyal

Combinatorial Reciprocity Theorems

Enumeration Done Geometrically



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Preface

Combinatorics is not a science, it's an attitude.
Mark Haiman

A combinatorial reciprocity theorem relates two classes of combinatorial objects via their counting functions. Consider a class $\mathcal{X}$ of combinatorial objects and let $f(n)$ be the function that counts the number of objects in $\mathcal{X}$ of size n, where "size" refers to some specific quantity that is naturally associated with the objects in $\mathcal{X}$. As in canonization, it requires two miracles for a combinatorial reciprocity to occur:

1. The function $f(n)$ is the restriction of some reasonable function (e.g., a polynomial) $F(x)$ to the positive integers, and
2. the evalutation $F(-n)$ is an integer of the same sign $\sigma \in \{\pm 1\}$ for all $n \in \mathbb{Z}_{>0}$.

In this situation it is human to ask if $\sigma F(-n)$ has a combinatorial meaning, that is, if there is a natural class $\mathcal{X}^\circ$ of combinatorial objects such that $\sigma F(-n)$ counts the objects of $\mathcal{X}^\circ$ of size n (where "size" again refers to some specific quantity, possibly quite different from the quantity we were measuring for $\mathcal{X}$). Combinatorial reciprocity theorems are among the most charming results in mathematics and, in contrast to canonization, can be found all over enumerative combinatorics and beyond.

As a first example consider the class of maps $[k] \to \mathbb{Z}_{>0}$ from a finite set $[k] = \{1, 2, \ldots, k\}$ into the positive integers and let $f(n) = n^k$ count the number of maps with range $[n]$. Thus $f(n)$ is the restriction of a polynomial and $(-1)^k f(-n) = n^k$ satisfies our second requirement above. This relates the number of maps $[k] \to [n]$ to itself. This relation is a genuine combinatorial reciprocity but the impression one is left with is that of being underwhelmed rather than charmed. Later in the book it will become clear that this example is not boring at all but for now let's try again.

The term "combinatorial reciprocity theorem" was coined by Richard Stanley in his 1974 paper [68] of the same title, in which he developed a firm standing of the subject. Stanley starts with a very appealing reciprocity that he attributes to Riordan: For a fixed integer d, let $f(n)$ count the number of d-subsets of an n-set, that is, the number of choices of taking d elements from a set of n elements without repetition. The counting function is given by the binomial coefficient

$$f(n) \;=\; \binom{n}{d} \;=\; \frac{1}{d!}\, n(n-1)\cdots(n-d+2)(n-d+1) \tag{0.1}$$

which is the restriction of a polynomial in n of degree d, and from the factorization we can read off that $(-1)^d f(-n)$ is a positive integer for every $n > 0$. More precisely,

$$(-1)^d f(-n) \;=\; \frac{1}{d!}\,(n+d-1)(n+d-2)\cdots(n+1)n \;=\; \binom{n+d-1}{d}$$

is the number of *d-multisubsets* of an n-set, that is, the number of picking d elements from $[n]$ with repetition. Now this is a combinatorial reciprocity! In formulas it reads

$$(-1)^d \binom{n}{d} \;=\; \binom{n+d-1}{d}. \tag{0.2}$$

This is a charming result in more than one way. It presents an intriguing connection between subsets and multisubsets via their counting functions, and its formal justification is completely within the realms of an undergraduate class in combinatorics. This result can be found in Riordan's book [57] on combinatorial analysis without further comment and, charmingly, Stanley states that his paper can be considered as "further comment". That further comment is necessary is apparent from the fact that the formal proof above falls short of explaining why these two sorts of objects are related by a combinatorial reciprocity. In particular, comparing coefficients in (0.2) cannot be the tool of choice for establishing more general reciprocity relations.

In this book we develop tools and techniques for handling combinatorial reciprocities. However, our own perspective is firmly rooted in *geometric* combinatorics and, thus, our emphasis is on the geometric nature of the combinatorial reciprocities. That is, for every class of combinatorial objects we associate a geometric object (such as a polytope or a polyhedral complex) in such a way that combinatorial features, including counting functions and reciprocity, are reflected in the geometry. In short, this book can be seen as *further comment with pictures*.

The prerequisites for this book are minimal: basic knowledge of linear algebra and combinatorics should suffice. The numerous exercises throughout the text are designed so that the book could easily be used for a graduate class in combinatorics.

Acknowledgments

Chapter 1

Four Polynomials

To many, mathematics is a collection of theorems. For me, mathematics is a collection of examples; a theorem is a statement about a collection of examples and the purpose of proving theorems is to classify and explain the examples...
John B. Conway

In the spirit of the above quote, this chapter serves as a source of examples and motivation for the theorems to come and the tools to be developed. Each of the following four sections introduces a family of examples together with a reciprocity statement which we will prove in later chapters.

1.1 Graph colorings

Graphs and their colorings are all-time favorites in introductory classes on discrete mathematics, and we too succumb to the temptation to start with one of the most beautiful examples. A **graph** $G = (V, E)$ is a discrete structure composed of a finite[1] set of **nodes** V and a collection $E \subseteq \binom{V}{2}$ of unordered pairs of nodes, called **edges**. More precisely, this defines a **simple** graph as it excludes the existence of multiple edges between nodes and, in particular, edges with equal endpoints, i.e., **loops**. We will, however, need such non-simple graphs in the sequel but we dread the formal overhead non-simple graphs entail and will trust in the reader's discretion to make the necessary modifications. The most charming feature of graphs is that they are easy to visualize and their natural habitat are the margins of textbooks or notepads. Figure 1.1 shows some examples.

[1] *Infinite* graphs are interesting in their own right; however, they are no fun to color-count and so will play no role in this book.

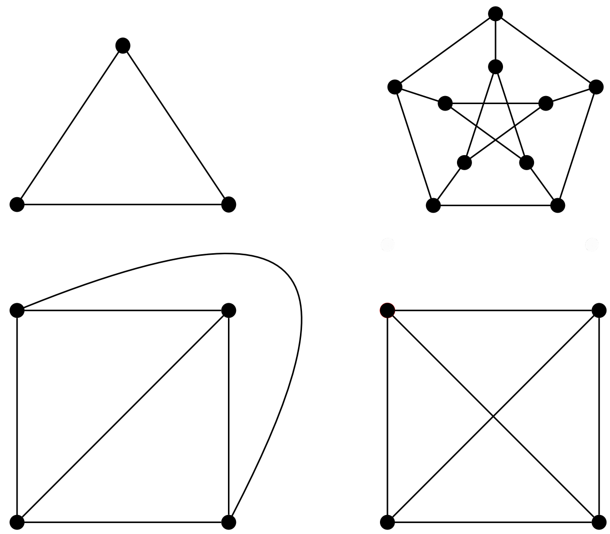


Fig. 1.1 Various graphs.

An n-**coloring** of a graph G is a map $c : V \to [n]$. An n-coloring c is called **proper** if no two nodes sharing an edge get assigned the same color, that is,

$$c(u) \neq c(v) \qquad \text{whenever} \qquad uv \in E\,.$$

The name *coloring* comes from the natural interpretation of thinking of $c(v)$ as one of n possible colors that we use for the node v. A proper coloring is one where adjacent nodes get different colors. Here is a first indication why considering simple graphs often suffices: the existence and even the number of n-colorings is unaffected by parallel edges, and there are simply no proper colorings in the presence of loops.

Much of the fame of colorings stems from a question that was asked around 1852 by Francis Guthrie and answered only some 124 years later. In order to state the question in modern terms, we call a graph G **planar** if G can be drawn in the plane (or scribbled in the margin) such that edges do not cross except possibly at nodes. For example, the last row in Figure 1.1 shows a planar and nonplanar embedding of the (planar) graph K_4. Here is Guthrie's famous conjecture, now a theorem:

Four-color theorem. *Every planar graph has a proper* 4*-coloring.*

There were several attempts at the four-color theorem before the first correct proof by Kenneth Appel and Wolfgang Haken. Here is one particularly interesting (but not yet successful) approach to proving the four-color theorem, due to George Birkhoff. For a (not necessarily planar) graph G, let

$$\chi_G(n) \ := \ \#\{c : V \to [n] \text{ proper } n\text{-coloring}\}\,.$$

The following observation, due to George Birkhoff and Hassler Whitney, is that $\chi_G(n)$ is the restriction to $\mathbb{Z}_{>0}$ of a particularly nice function:

Proposition 1.1. *Let $G = (V, E)$ be a loopless graph. Then $\chi_G(n)$ agrees with a polynomial of degree $|V|$ with integral coefficients.*

By a slight abuse of notation, we identify $\chi_G(n)$ with this polynomial and call it the **chromatic polynomial** of G. Nevertheless, let's emphasize that, so far, only the values of $\chi_G(n)$ at positive integral arguments have an interpretation in terms of G.

One proof of Proposition 1.1 is interesting in its own right, as it exemplifies *deletion–contraction* arguments which we will revisit in Chapter 7 For $e \in E$, the **deletion** of e results in the graph $G \setminus e := (V, E \setminus \{e\})$. The **contraction** G/e is the graph obtained by identifying the two nodes incident to e and removing any newly created parallel edges. An example is given in Figure 1.2.

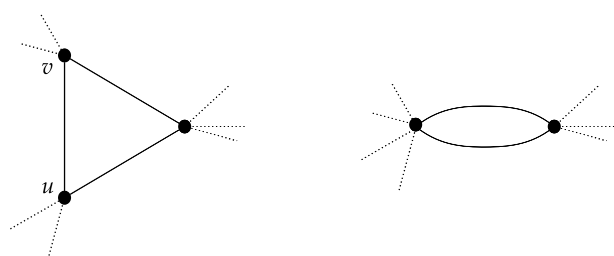


Fig. 1.2 Contracting the edge $e = uv$.

Proof of Proposition 1.1. We induct on $|E|$. For $|E| = 0$ there are no coloring restrictions and $\chi_G(n) = n^{|V|}$. One step further, assume that G has a single edge $e = uv$. Then we can color all nodes $V \setminus u$ arbitrarily and, assuming $n \geq 2$, can color u with any color $\neq c(v)$. Thus, the chromatic polynomial is $\chi_G(n) = n^{d-1}(n-1)$ where $d = |V|$. For the induction step now let $e = uv \in E$. We claim

$$\chi_G(n) \ = \ \chi_{G\setminus e}(n) \ - \ \chi_{G/e}(n) \tag{1.1}$$

Indeed, a coloring c of $G \setminus e$ fails to be a coloring of G if $c(u) = c(v)$. That is, we are over-counting by all proper colorings that assign the same color to u and v. These are precisely the proper n-colorings of G/e. By (1.1) and the induction hypothesis, $\chi_G(n)$ is the difference of a polynomial of degree $d = |V|$ and a polynomial of degree $d-1$, both with integer coefficients. $\square$

The above proof and, more precisely, the deletion–contraction relation (1.1) reveal more about chromatic polynomials, which we invite the reader to show in Exercise 1.23:

Corollary 1.2. *Let G be a loopless graph on $d \geq 1$ nodes and $\chi_G(n) = c_d n^d + c_{d-1}n^{d-1} + \cdots + c_0$ its chromatic polynomial. Then*

(a) *the leading coefficient $c_d = 1$;*

(b) *the constant coefficient $c_0 = 0$;*

(c) $(-1)^d \chi_G(-n) > 0$ *for all integral $n \geq 1$.*

In particular the last property prompts the following natural question that we alluded to in the preface and that lies at the heart of this book.

Question: Do the evaluations $(-1)^{|V|}\chi_G(-n)$ have combinatorial meaning?

This question was first asked (and beautifully answered) by Richard Stanley in 1973. To reproduce his answer, we need the notion of orientations on graphs. Again, to keep the formal pain level at a minimum, let's denote the nodes of G by $v_1, v_2, \dots, v_d$. We define an **orientation** on G through a subset $\rho \subseteq E$; for an edge $e = v_i v_j \in E$ with $i < j$ we direct

$$v_i \xleftarrow{e} v_j \ \text{ if } \ e \in \rho \qquad \text{and} \qquad v_i \xrightarrow{e} v_j \ \text{ if } \ e \notin \rho\,.$$

We denote the oriented graph by ${}_\rho G$ and will sometimes write ${}_\rho G = (V, E, \rho)$. Said differently, we may think of G as canonically oriented by directing edges from small index to large index and ρ records the edges on which this orientation is reversed; see Figure 1.3 for an example.

A **directed path** is a constellation $v_{i_0} \to v_{i_1} \to \cdots \to v_{i_s}$ in ${}_\rho G$ such that $v_{i_k} \neq v_{i_l}$ for $0 \leq k < l \leq s$, and it is called a **directed cycle** if $v_{i_0} = v_{i_s}$. An orientation ρ of G is **acyclic** if there are no directed cycles in ${}_\rho G$.

Here is the connection between proper colorings and acyclic orientations: Given a proper coloring c, we define the orientation

$$\rho := \left\{ v_i v_j \in E : i < j,\ c(v_i) > c(v_j) \right\}.$$

That is, the edge from lower index i to higher index j is directed along its **color gradient** $c(v_j) - c(v_i)$. We call this orientation ρ **induced** by the

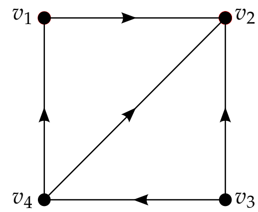


Fig. 1.3 An orientation given by $\rho = \{14, 23, 24\}$.

coloring c. For example, the orientation pictured in Figure 1.3 is induced by the coloring shown in Figure 1.4.

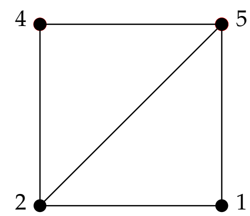


Fig. 1.4 A coloring that induces the orientation in Figure 1.3.

Proposition 1.3. *Let $c : V \to [n]$ be a proper coloring and ρ the induced orientation on G. Then ${}_\rho G$ is acyclic.*

Proof. Assume that $v_{i_0} \to v_{i_1} \to \cdots \to v_{i_s} \to v_{i_0}$ is a directed cycle in ${}_\rho G$. Then $c(v_{i_0}) > c(v_{i_1}) > \cdots > c(v_{i_s}) > c(v_{i_0})$ which is a contradiction. $\square$

As there are only finitely many acyclic orientations on G, we might count colorings according to the acyclic orientation they induce. An orientation ρ and a n-coloring c of G are called **compatible** if for every oriented edge $u \to v$ in ${}_\rho G$ we have $c(u) \geq c(v)$. The pair (ρ, c) is called **strictly** compatible if $c(u) > c(v)$ for every oriented edge $u \to v$.

Proposition 1.4. *If (ρ, c) is strictly compatible, then c is a proper coloring and ρ is an acyclic orientation on G. In particular, $\chi_G(n)$ is the number of strictly compatible pairs (ρ, c) where c is a proper n-coloring.*

Proof. If (ρ, c) are strictly compatible then, since every edge is oriented, $c(u) > c(v)$ or $c(u) < c(v)$ whenever $uv \in E$. Hence c is a proper coloring and ρ is exactly the orientation induced by c. Acyclicity of ${}_\rho G$ now follows from Proposition 1.3. $\square$

We are finally ready for our first combinatorial reciprocity theorem.

Theorem 1.5 (Stanley). *Let G be a finite graph on d nodes and $\chi_G(n)$ its chromatic polynomial. Then $(-1)^d \chi_G(-n)$ is the number of compatible pairs (ρ, c) where c is a n-coloring and ρ is an acyclic orientation. In particular, $(-1)^d \chi_G(-1)$ is the number of acyclic orientations of G.*

As one illustration of this theorem, consider the graph G in Figure 1.5; its chromatic polynomial is $\chi_G(n) = n(n-1)(n-2)^2$, and so Theorem 1.5 suggests that G should admit 18 acyclic orientations. Indeed, there are 6 acylic orientations of the subgraph formed by v_1, v_2, and v_4, and for the remaining two edges, one of the four possible combined orientations of v_2v_3 and v_3v_4 produces a cycle with v_2v_4, so there are a total of $6 \cdot 3 = 18$ acyclic orientations.

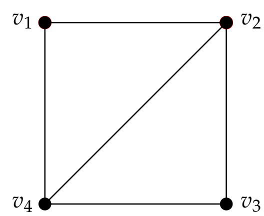


Fig. 1.5 This graph has 18 acyclic orientations.

A deletion–contraction proof of Theorem 1.5 is outlined in Exercise 1.26; we will give a geometric proof in Section 7.3.

1.2 Flows on graphs

Given a graph $G = (V, E)$ together with an orientation ρ and a finite abelian group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, a $\mathbb{Z}_n$-**flow** is a map $f : E \to \mathbb{Z}_n$ that assigns a value $f(e) \in \mathbb{Z}_n$ to every edge $e \in E$ such that there is conservation of flow at every node v:

$$\sum_{\xrightarrow{e} v} f(e) \;=\; \sum_{v \xrightarrow{e}} f(e) \,,$$

that is, what "flows" into the node v is precisely what "flows" out of v. This physical interpretation is a bit shaky as the commodity flowing along edges are elements of $\mathbb{Z}_n$, and the flow conservation is with respect to the group structure. The set

$$\operatorname{supp}(f) := \{e \in E \,:\, f(e) \neq 0\}$$

is the **support** of f, and a $\mathbb{Z}_n$-flow f is **nowhere zero** if $\operatorname{supp}(f) = E$. In this section we will be concerned with counting nowhere-zero $\mathbb{Z}_n$-flows, and so we define

$$\varphi_G(n) \;:=\; \#\big\{f \text{ nowhere-zero } \mathbb{Z}_n\text{-flow on } {}_\rho G\big\}\,.$$

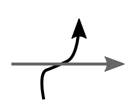
A priory, the counting function $\varphi_G(n)$ depends on our chosen orientation ρ, but our language suggests that this is not the case, which we invite the reader to verify in Exercise 1.28:

Proposition 1.6. *The flow-counting function $\varphi_G(n)$ is independent on the orientation ρ of G.*

A **connected component** of the graph G is a maximal subgraph of G in which any two nodes are connected by a path. A graph G is **connected** if it has only one connected component.[2] As the reader will discover (at the latest when working on Exercises 1.30 and 1.31), G will not have any nowhere-zero flow if G has a **bridge**, that is, an edge whose removal increases the number of connected components of G.

To motivate why we care about counting nowhere-zero flows, let's assume that G is a *planar* bridgeless graph with a given embedding into the plane. The drawing of G subdivides the plane into connected regions in which two points lie in the same region whenever they can be joined by a path in $\mathbb{R}^2$ that does not meet G. Two such regions are neighboring if their topological closures share a proper (i.e., 1-dimensional) part of their boundaries. This induces a graph structure on the subdivision of the plane: For the given embedding of G, we define the **dual graph** G^* as the graph with nodes corresponding to the regions and two regions C_1, C_2 share an edge e^* if an original edge e is properly contained in both their boundaries. As we can see in the example pictured in Figure 1.6, G^* is typically a non-simple graph with parallel edges. If G had bridges, G^* would have loops.

Given an orientation of G, an orientation on G^* is induced by, for example, rotating the edge clockwise. That is, the dual edge will "point" east assuming that the primal edge "points" north.



By carefully adding G^* to the picture it can be seen that dualizing G^* recovers G, i.e., $(G^*)^* = G$.

[2] These notions refer to an *unoriented* graph.

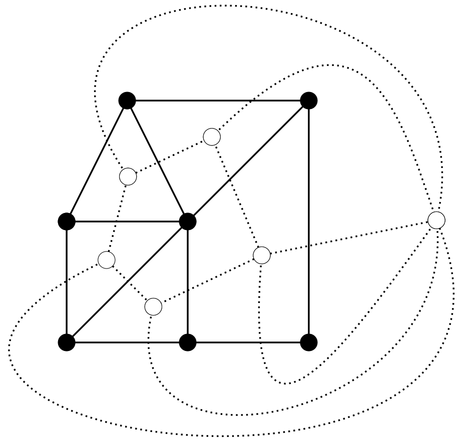


Fig. 1.6 A graph and its dual.

Our interest in flows lies in the connection to colorings: Let c be a n-coloring of G and for a change let's assume that c takes on colors in $\mathbb{Z}_n$. After giving G an orientation, we can record the color-gradients $t(uv) = c(v) - c(u)$ for every oriented edge $u \to v$ and, knowing the color of a single node v_0, we can recover the coloring from $t : E \to \mathbb{Z}_n$: For a node $v \in V$ simply choose an undirected path $v_0 = p_0p_1p_2 \cdots p_k = v$ from v_0 to v. Then while walking along this path we can color every node p_i by adding or subtracting $t(p_{i-1}p_i)$ to $c(p_{i-1})$ depending whether we walked the edge $p_{i-1}p_i$ with or against its orientation. This process is illustrated in Figure 1.7. The color $c(v)$ is independent of the chosen path and thus, walking along a cycle in G the sum of the values $t(e)$ of edges along their orientation minus those against their orientation has to be zero. Now, via the correspondence of primal and dual edges, t induces a map $f : E^* \to \mathbb{Z}_n$ on the dual graph G^*. Every node of G^* represents a region that is bounded by a cycle in G, and the orientation on G^* is such that walking around this cycle clockwise, each edge traversed along its orientation corresponds to a dual edge into the region while counter-clockwise edges dually point out of the region. The cycle condition above then proves:

Proposition 1.7. *Let G be a connected planar graph with dual G^*. For every n-coloring c of G, the induced map f is a $\mathbb{Z}_n$-flow on G^* and every such flow arises this way. Moreover, the coloring c is proper if and only if f is nowhere zero.*

Conversely, for a given (nowhere-zero) flow f on G^* one can construct a (proper) coloring on G (see Exercise 1.29). In light of all this, we can rephrase the four-color theorem as follows:

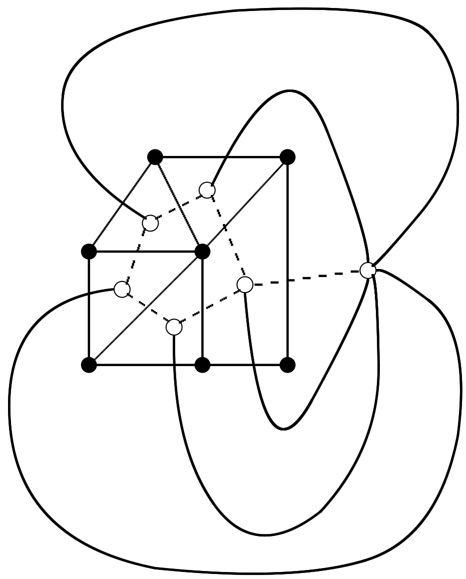


Fig. 1.7 Passage from colorings to dual flows and back.

Corollary 1.8 (Dual four-color theorem). *If G is a planar bridgeless graph, then $\varphi_G(4) > 0$.*

This perspective to colorings of planar graphs was pioneered by William Tutte who initiated the study of $\varphi_G(n)$ for all (not necessary planar) graphs. To see how much flows differ from colorings, observe that there is no universal constant n_0 such that every graph has a proper n_0-coloring. The analagous statement for flows is not so clear and, in fact, Tutte conjectured the following:

5-flow Conjecture. Every bridgeless graph has a nowhere-zero $\mathbb{Z}_5$-flow.

This sounds like a rather daring conjecture as it is not even clear that there is any such n such that every bridgeless graph has a nowhere-zero $\mathbb{Z}_n$-flow. However, it was shown by Paul Seymour that $n \leq 6$ works. In Exercise 1.34 you will show that there exist graphs that do not admit a nowhere-zero $\mathbb{Z}_4$-flow. So, similar to the history of the four-color theorem, the gap between the conjectured and the actual truth could not be smaller.

On the enumerative side, we have the following:

Proposition 1.9. *If G is a bridgeless connected graph, then $\varphi_G(n)$ agrees with a polynomial with integer coefficients of degree $|E| - |V| + 1$ and leading coefficient* 1*.*

Again, we will abuse notation and refer to $\varphi_G(n)$ as the **flow polynomial**. The proof of the polynomiality is a deletion–contraction argument that is deferred to Exercise 1.30.

Towards a reciprocity statement, we need a notion dual to acyclic orientations: an orientation ρ on G is **totally cyclic** if every edge in ${}_\rho G$ is contained in a directed cycle. Let just quickly define the **cyclotomic number** of G as $\xi(G) := |E| - |V| + c$ where c is the number of connected components of G.

Theorem 1.10. *Let G be a bridgeless graph. For every positive integer n, the evaluation $(-1)^{\xi(G)}\varphi_G(-n)$ counts the number of pairs (f,ρ) where f is a $\mathbb{Z}_n$-flow and ρ is a totally-cyclic reorientation of $G/\operatorname{supp}(f)$. In particular, $(-1)^{\xi(G)}\varphi_G(0)$ is the number of totally-cyclic orientations of G.*

We will prove this theorem in Section 7.4.

1.3 Order polynomials

A **partially ordered set**, or **poset** for short, is a set Π together with a binary relation $\preceq_\Pi$ that is

reflexive: $a \preceq_\Pi a$,

transitive: $a \preceq_\Pi b \preceq_\Pi c$ implies $a \preceq_\Pi c$, and

anti-symmetric: $a \preceq_\Pi b$ and $b \preceq_\Pi a$ implies $a = b$,

for all $a, b, c \in \Pi$. We write $\preceq$ if the poset is clear from the context.

Partially ordered sets are ubiquitous structures in combinatorics and, as we will amply demonstrate soon, are indispensable in enumerative and geometric combinatorics. The essence of a poset is encoded by its **cover relations**: an element $a \in \Pi$ is covered by an element b if

$$[a,b] := \{z \in \Pi \,:\, a \preceq z \preceq b\} = \{a,b\},$$

in plain English: $a \preceq b$ and there is nothing "between" a and b. From its cover relations we can recover the poset by taking the transitive closure and adding in the reflexive relations. The cover relations can be thought of as a directed graph, and this gives an effective way to picture a poset: The

Hasse diagram of Π is a drawing of the directed graph of cover relations in Π as an (undirected) graph where the node a is drawn lower than the node b whenever $a \prec b$. Here is an example: For $n \in \mathbb{Z}_{>0}$ we define D_n as the set $[n] = \{1, 2, \dots, n\}$ ordered by divisibilty, that is, $a \preceq b$ if a divides b. The Hasse diagram of D_{10} is given in Figure 1.8.

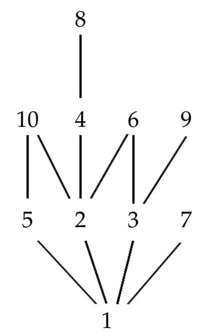


Fig. 1.8 D_{10}: The set $[10]$, partially ordered by divisibility.

This example truly is a *partial* order as, for example, 2 and 7 are not comparable. A poset in which every element is comparable to every other element is a **chain**. To be more precise: the poset Π is a chain if we have either $a \preceq b$ or $b \preceq a$ for any two elements $a, b \in \Pi$. The elements of a chain are **totally** or **linearly ordered**.

A map $\phi : \Pi \to \Pi'$ is **(weakly) order preserving** if for all $a, b \in \Pi$

$$a \preceq_\Pi b \quad \Longrightarrow \quad \phi(a) \preceq_{\Pi'} \phi(b)$$

and **strictly order preserving** if

$$a \prec_\Pi b \quad \Longrightarrow \quad \phi(a) \prec_{\Pi'} \phi(b) .$$

For example, we can label the elements of a chain Π such that

$$\Pi = \{a_1 \prec a_2 \prec \cdots \prec a_n\}$$

which makes Π **isomorphic** to $[n] := \{1 < 2 < \cdots < n\}$, in the sense that there is an order preserving bijection between Π and $[n]$.

Order preserving maps are the natural morphisms (even in a categorical sense) between posets, and in this section we will be concerned with counting (strictly) order preserving maps from a poset into chains. A strictly order preserving map ϕ from one chain $[d]$ into another $[n]$ exists only if $d \leq n$ and is then determined by

$$1 \le \phi(1) < \phi(2) < \cdots < \phi(d) \le n .$$

Thus, the number of such maps equals $\binom{n}{d}$, the number of d-subsets of an n-set. In the case of a general poset Π, we define the **strict order polynomial**

$$\Omega^{\circ}_{\Pi}(n) \;:=\; \#\{\phi : \Pi \to [n] \text{ strictly order preserving}\} .$$

As we have just seen, $\Omega^{\circ}_{\Pi}(n)$ is indeed a polynomial when $\Pi = [d]$. We now show that polynomiality holds for all posets Π:

Proposition 1.11. *For a finite poset Π, the function $\Omega^{\circ}_{\Pi}(n)$ agrees with a polynomial of degree $|\Pi|$ with rational coefficients.*

Proof. Let $d := |\Pi|$ and $\phi : \Pi \to [n]$ be a strictly order preserving map. Now ϕ factors uniquely

$$\begin{tikzcd} \Pi \arrow[rr, "\phi"] \arrow[dr, two heads, "\sigma"'] & & {[n]} \\ & \phi(\Pi) \arrow[ur, hook, "\iota"'] & \end{tikzcd}$$

into a surjective map σ onto $\phi(\Pi)$ followed by an injection ι. (Use the functions $\sigma(a) := \phi(a)$ and $\iota(a) := a$, defined with domains and codomains pictured above.) The image $\phi(\Pi)$ is a subposet of a chain and so is itself a chain. Thus $\Omega^{\circ}_{\Pi}(n)$ counts the number of pairs (σ, ι) of strictly order preserving maps $\Pi \twoheadrightarrow [r] \rightarrowtail [n]$ for $r = 1, 2, \dots, d$. For fixed r, there are only finitely many order preserving surjections $\sigma : \Pi \to [r]$, say, s_r many. As we discussed earlier, the number of strictly order preserving maps $[r] \to [n]$ is exactly $\binom{n}{r}$, which is a rational polynomial in n of degree r. Hence, for fixed r, there are $s_r\binom{n}{r}$ many pairs (σ, ι) and we obtain

$$\Omega^{\circ}_{\Pi}(n) \;=\; s_d\binom{n}{d} \;+\; s_{d-1}\binom{n}{d-1} \;+\; \cdots \;+\; s_1\binom{n}{1},$$

which finishes our proof. $\square$

As an aside, let's mention that the above proposition proves that $\Omega^{\circ}_{\Pi}(n)$ is a polynomial with *integral* coefficients if we use $\left\{\binom{n}{r} \,:\, r \in \mathbb{Z}_{\ge 0}\right\}$ as a basis for $\mathbb{R}[n]$. That the binomial coefficients indeed form a basis for the univariate polynomials follows from the proposition above: If Π is an **antichain** on d elements, i.e., a poset in which no elements are related,

$$\Omega^{\circ}_{\Pi}(n) \;=\; n^d \;=\; s_d\binom{n}{d} \;+\; s_{d-1}\binom{n}{d-1} \;+\; \cdots \;+\; s_1\binom{d}{1}.$$

In this case, the coefficients $s_r = S(d,r)$ are the **Stirling numbers of the second kind** which count the number of surjective maps $[d] \twoheadrightarrow [r]$. (The Stirling numbers might come in handy in Exercise 1.27.)

For the case that Π is a d-chain, the reciprocity statement (0.2) says that $(-1)^d \Omega^\circ_\Pi(-n)$ gives the number of d-multisubsets of an n-set, which equals, in turn, the number of (weak) order preserving maps from a d-chain to an n-chain. Our next reciprocity theorem expresses this duality between weak and strict order preserving maps from a general poset into chains. You can already guess what is coming. We define the **order polynomial**

$$\Omega_\Pi(n) \;=\; \#\{\phi : \Pi \to [n] \text{ order preserving}\}\,.$$

A slight modification (which we invite the reader to check in Exercise 1.36) of our proof of Proposition 1.11 implies that $\Omega_\Pi(n)$ indeed agrees with polynomial in n, and the following reciprocity theorem gives the relationship between the two polynomials $\Omega_\Pi(n)$ and $\Omega^\circ_\Pi(n)$.

Theorem 1.12. *Let Π be a finite poset. Then*

$$(-1)^{|\Pi|}\,\Omega^\circ_\Pi(-n) \;=\; \Omega_\Pi(n)\,.$$

We will prove this theorem in Chapter 2. To further motivate the study of order polynomials, we remark that a poset Π gives rise to an oriented graph by way of the cover relations of Π. Conversely, the binary relation given by an oriented graph G can be completed to a partial order $\Pi(G)$ by adding the necessary transitive and reflexive relations if and only if G is acyclic. The following result will be the subject of Exercise 1.35.

Proposition 1.13. *Let ${}_\rho G = (V, E, \rho)$ be an acyclic graph and $\Pi = \Pi({}_\rho G)$ the induced poset. A map $c : V \to [n]$ is strictly compatible with the orientation ρ of G if and only if c is a strictly order preserving map $\Pi \to [n]$.*

In Proposition 1.4 we identified the number of n-colorings $\chi_G(n)$ of G as the number of colorings c strictly compatible with some acyclic orientation ρ of G, and so this proves:

Corollary 1.14. *The chromatic polynomial $\chi_G(n)$ of a graph G is the sum of the order polynomials $\Omega^\circ_{\Pi({}_\rho G)}(n)$ for all acyclic orientations ρ of G.*

1.4 Ehrhart polynomials

The formulation of (0.1) in terms of d-subsets of a n-set has a straightforward geometric interpretation that will fuel most of what is about to come: The d-subsets of $[n]$ correspond precisely to the points with integral coordinates in the set

$$(n+1)\Delta_d^\circ = \left\{(x_1,\dots,x_d) \in \mathbb{R}^d \,:\, 0 < x_1 < x_2 < \cdots < x_d < n+1\right\}. \qquad (1.2)$$

Let's explain the notation on the left-hand side: we define

$$\Delta_d^\circ := \left\{(x_1,\dots,x_d) \in \mathbb{R}^d \,:\, 0 < x_1 < x_2 < \cdots < x_d < 1\right\},$$

and for a set $S \subseteq \mathbb{R}^d$ and a positive integer n, we define

$$n\,S := \{n\,\mathbf{x} \,:\, \mathbf{x} \in S\},$$

the **n-th dilate** of S. (We hope the notation in (1.2) now makes sense.) For example, when $d = 2$,

$$\Delta_2^\circ = \left\{(x_1,x_2) \in \mathbb{R}^2 \,:\, 0 < x_1 < x_2 < 1\right\}$$

is the interior of a triangle, and every integer point (x_1,x_2) in the $(n+1)$-st dilate of Δ_2° satisfies $0 < x_1 < x_2 < n+1$ or, equivalently, $1 \le x_1 < x_2 \le n$. We illustrate these integer points for the case $n = 6$ in Figure 1.9.

A **convex lattice polygon** $\mathcal{P} \subset \mathbb{R}^2$ is the smallest convex set containing a given finite set of non-collinear integer points in the plane. The **interior** of $\mathcal{P}$ is denoted by $\mathcal{P}^\circ$. Convex polygons are 2-dimensional instances of **convex polytopes**, which live in any dimension and whose properties we will study in detail in Chapter 3. For now, we count on the reader's intuition about objects such as vertices and edges of a polygon, which will be defined rigorously later on.

For a bounded set $S \subset \mathbb{R}^2$, we write $E(S) := \#(S \cap \mathbb{Z}^2)$ for the number of lattice points in S. Our example above motivates the definitions of the counting functions

$$\mathrm{ehr}_{\mathcal{P}^\circ}(n) := E(n\mathcal{P}^\circ) = \#\left(n\mathcal{P}^\circ \cap \mathbb{Z}^2\right)$$

and

$$\mathrm{ehr}_{\mathcal{P}}(n) := E(n\mathcal{P}) = \#\left(n\mathcal{P} \cap \mathbb{Z}^2\right),$$

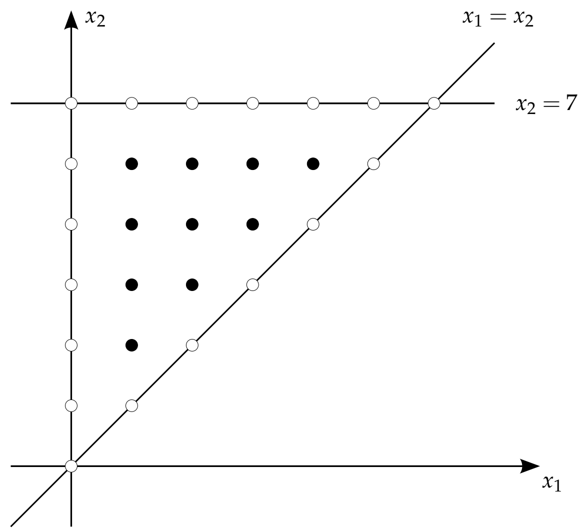


Fig. 1.9 The integer points in $7\Delta_2^\circ$.

called the **Ehrhart functions** of $\mathcal{P}$, for reasons that will be explained in Section 4.5.

As we know from (0.1), the number of integer lattice points in the $(n+1)$-st dilate of Δ_2° is given by the polynomial

$$\mathrm{ehr}_{\Delta_2^\circ}(n+1) = \binom{n}{2}.$$

To make the combinatorial reciprocity statement given by (0.1) geometric, we observe that the number of weak order preserving maps from $[n]$ into $[2]$ is given by the integer points in the $(n-1)$-st dilate of

$$\Delta_2 \;=\; \left\{(x_1, x_2) \in \mathbb{R}^2 \,:\, 0 \le x_1 \le x_2 \le 1\right\},$$

the closure of Δ_2°. The combinatorial reciprocity statement given by (0.1) now reads $(-1)^2\binom{-n}{2}$ *equals the number of integer points in* $(n-1)\Delta_2$. Unraveling the parameters (and making appropriate shifts), we can rephrase this as: $(-1)^2\,\mathrm{ehr}_{\Delta_2^\circ}(-n)$ equals the number of integer points in $n\Delta_2$. The reciprocity theorem featured in this section states that this holds for all convex lattice polygons; in Section 4.5 we will prove an analogue in all dimensions.

Theorem 1.15. *Let $\mathcal{P} \subset \mathbb{R}^2$ be a lattice polygon. Then $\mathrm{ehr}_{\mathcal{P}}(n)$ agrees with a polynomial of degree 2 with rational coefficients, and $(-1)^2\,\mathrm{ehr}_{\mathcal{P}}(-n)$ equals the number of integer points in the interior $n\mathcal{P}^\circ$.*

In the remainder of this section we will prove this theorem. The proof will be a series of simplifying steps that are similar in spirit to those that we will employ for the general result in Section 4.5.

As a first step, we reduce the problem of showing polynomiality for Ehrhart functions of arbitrary lattice polygons to that of lattice *triangles*. Let $\mathcal{P}$ be a lattice polygon in the plane with n vertices. We can **triangulate** $\mathcal{P}$ by cutting the polygon along sufficiently many (exactly $n-3$) non-intersecting diagonals, as in Figure 1.10. The result is a set of $n-2$ lattice triangles that cover $\mathcal{P}$. We denote by $\mathcal{T}$ the collection of **faces** of all these triangles, that is, $\mathcal{T}$ consists of n zero-dimensional polytopes (vertices), $2n-3$ one-dimensional polytopes (edges), and $n-2$ two-dimensional polytopes (triangles).

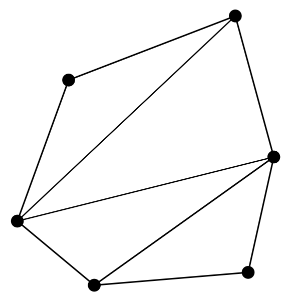


Fig. 1.10 A triangulation of a hexagon.

Our triangulation is a well-behaved collection of polytopes in the plane in the sense that they intersect nicely: if two elements of $\mathcal{T}$ intersect, then they intersect in a common face of both. This is useful, as counting lattice points is a *valuation*:[3] For $S, T \subset \mathbb{R}^2$,

$$E(S \cup T) \;=\; E(S) + E(T) - E(S \cap T)\,, \tag{1.3}$$

and applying the inclusion-exclusion relation (1.3) repeatedly to the elements in our triangulation of $\mathcal{P}$ yields

$$\operatorname{ehr}_{\mathcal{P}}(n) \;=\; \sum_{\mathcal{F} \in \mathcal{T}} \mu_{\mathcal{T}}(\mathcal{F}) \operatorname{ehr}_{\mathcal{F}}(n)\,, \tag{1.4}$$

where the $\mu_{\mathcal{T}}(\mathcal{F})$'s are some coefficients that correct for over-counting. If $\mathcal{F}$ is a triangle, then $\mu_{\mathcal{T}}(\mathcal{F}) = 1$—after all, we want to count the lattice points in $\mathcal{P}$ which is covered by the triangles. For an edge $\mathcal{F}$ of the triangulation, we have to make the following distinction: $\mathcal{F}$ is an **interior edge** of $\mathcal{T}$ if it is contained in two triangles. In this cases the lattice points in $\mathcal{F}$ get counted

[3] We'll have more to say about valuations in Section 3.2.

twice and in order to compensate for this, we set $\mu_{\mathcal{T}}(\mathcal{F}) = -1$. In the case that $\mathcal{F}$ is a **boundary edge**, i.e., $\mathcal{F}$ lies in only one triangle of $\mathcal{T}$, there is no over-counting and we can set $\mu_{\mathcal{T}}(\mathcal{F}) = 0$. To generalize this to all faces of $\mathcal{T}$, let's call a face $\mathcal{F} \in \mathcal{T}$ a **boundary face** of $\mathcal{T}$ if $\mathcal{F} \subset \partial\mathcal{P}$ and an **interior face** otherwise. We can give the coefficients $\mu_{\mathcal{T}}(\mathcal{F})$ explicitly as follows.

Proposition 1.16. *Let $\mathcal{T}$ be a triangulation of a lattice polygon $\mathcal{P} \subset \mathbb{R}^2$. Then the coefficients $\mu_{\mathcal{T}}(\mathcal{F})$ in (1.4) are given by*

$$\mu_{\mathcal{T}}(\mathcal{F}) \;=\; \begin{cases} (-1)^{2-\dim\mathcal{F}} & \text{if } \mathcal{F} \text{ is interior,} \\ 0 & \text{otherwise.} \end{cases}$$

For boundary vertices $\mathcal{F} = \{v\}$, we can check that $\mu_{\mathcal{T}}(\mathcal{F}) = 0$ is correct: the vertex is counted positively as a lattice point by every incident triangle and negatively by every incident interior edge. As there are exactly one interior edge less than incident triangles, we do not count the vertex more than once. For an interior vertex, the number of incident triangles and incident (interior) edges are equal and hence $\mu_{\mathcal{T}}(\mathcal{F}) = 1$. In triangulations of $\mathcal{P}$ obtained by cutting along diagonals we never encounter *interior* vertices, however, they will appear soon when we consider a different type of triangulation.

The coefficient $\mu_{\mathcal{T}}(\mathcal{F})$ for a triangulation of a polygon was easy to argue and to verify in the plane. For higher-dimensional polytopes we will have to resort to more algebraic and geometric means. The right algebraic setup will be discussed in Chapter 2 where we will make use of the fact that a triangulation $\mathcal{T}$ constitutes a partially ordered set. In the language of posets, $\mu_{\mathcal{T}}(\mathcal{F})$ is an evaluation of the *Möbius function* for the poset $\mathcal{T}$. Möbius functions are esthetically satisfying but are in general difficult to compute. However, we are dealing with situations with plenty of geometry involved and we will make use of that in Chapter 5 to give a statement analogous to Proposition 1.16 in general dimension.

Returning to our 2-dimensional setting, showing that $\mathrm{ehr}_{\mathcal{F}}(n)$ is a polynomial whenever $\mathcal{F}$ is a lattice point, a lattice segment, or a lattice triangle gives us the first half of Theorem 1.15. If $\mathcal{F}$ is a vertex, then $\mathrm{ehr}_{\mathcal{F}}(n) = 1$. If $\mathcal{F} \in \mathcal{T}$ is an edge of one of the triangles and thus a lattice segment, verifying that $\mathrm{ehr}_{\mathcal{F}}(n)$ is a polynomial is the content of Exercise 1.37.

The last challenge now is the reciprocity for lattice triangles. For the rest of this section, let $\triangle \subset \mathbb{R}^2$ be a fixed lattice triangle in the plane. The idea that we will use is to triangulate the dilates $n\triangle$ for $n \geq 1$, but the triangulation will change with n. Figure 1.11 gives the picture for $n = 1,2,3$.

We trust that the reader can imagine the triangulation for all values of n. The special property of this triangulation is that up to lattice translations,

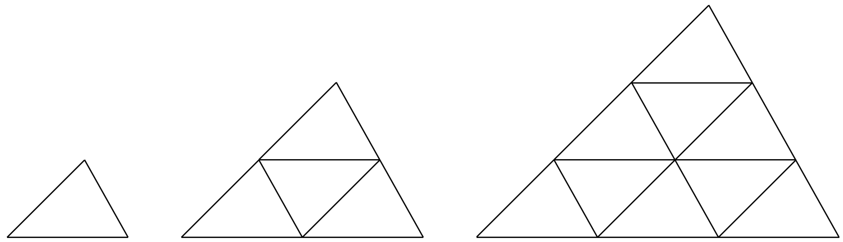


Fig. 1.11 Special triangulations of a lattice triangle.

there are only very few different pieces. In fact, there are only two different lattice triangles used in the triangulation of $n\triangle$: there is $\triangle$ itself and (lattice translates of) the reflection of $\triangle$ in the origin, which we will denote by $\triangledown$. As for edges, we have three different kinds of edges, namely the edges $\diagup$, $-$, and $\diagdown$. Up to lattice translation, there is only one vertex $\bullet$.

Now let's count how many copies of each tile occur in these special triangulations; let $t(\mathcal{P},n)$ denote the number of times $\mathcal{P}$ appears in our triangulation of $n\triangle$. As for triangles, we count

$$t(\triangle,n) \;=\; \binom{n+1}{2} \quad \text{and} \quad t(\triangledown,n) \;=\; \binom{n}{2}.$$

For the interior edges, observe that each interior edge is incident to a unique upside-down triangle $\triangledown$ and consequently

$$t(\diagup,n) \;=\; t(-,n) \;=\; t(\diagdown,n) \;=\; \binom{n}{2}.$$

Similarly, for interior vertices, we get

$$t(\bullet,n) \;=\; \binom{n-1}{2}.$$

Thus with (1.4), the Ehrhart function for the triangle $\triangle$ is

$$\begin{aligned} \mathrm{ehr}_{\triangle}(n) \;=\; & \binom{n+1}{2}E(\triangle) + \binom{n}{2}E(\triangledown) \\ & - \binom{n}{2}\big(E(\diagup) + E(-) + E(\diagdown)\big) \\ & + \binom{n-1}{2}E(\bullet)\,. \end{aligned} \tag{1.5}$$

This proves that $\mathrm{ehr}_{\triangle}(n)$ agrees with a polynomial of degree 2, and together with (1.4) this establishes the first half of Theorem 1.15.

To prove the combinatorial reciprocity of Ehrhart polynomials in the plane, let's make the following useful observation.

Proposition 1.17. *If for every lattice polygon $\mathcal{P} \subset \mathbb{R}^2$ we have that $\mathrm{ehr}_{\mathcal{P}}(-1)$ equals the number of lattice points in the interior of $\mathcal{P}$, then $\mathrm{ehr}_{\mathcal{P}}(-n) = E(n\mathcal{P}^\circ)$ for all $n \geq 1$.*

Proof. For fixed $n \geq 1$, let's denote by $\mathcal{Q}$ the lattice polygon $n\mathcal{P}$. We see that $\mathrm{ehr}_{\mathcal{Q}}(m) = E(m(n\mathcal{P}))$ for all $m \geq 1$. Hence the Ehrhart polynomial of $\mathcal{Q}$ is given by $\mathrm{ehr}_{\mathcal{P}}(mn)$ and for $m = -1$ we conclude

$$\mathrm{ehr}_{\mathcal{P}}(-n) = \mathrm{ehr}_{\mathcal{Q}}(-1) = E(\mathcal{Q}^\circ) = E(n\mathcal{P}^\circ)$$

which finishes our proof. □

To establish the combinatorial reciprocity of Theorem 1.15 for triangles, we can simply substitute $n = -1$ into (1.5) and use (0.2) to obtain

$$\mathrm{ehr}_{\triangle}(-1) \;=\; E(\triangledown) - E(\diagup) - E(-) - E(\diagdown) + 3E(\bullet),$$

which is exactly the number of interior lattice points of $\triangledown$. Observing that $\triangle$ and $\triangledown$ have the same number of lattice points finishes the argument.

For the general case, Exercise 1.37 gives

$$\mathrm{ehr}_{\mathcal{P}}(-1) \;=\; \sum_{\mathcal{F} \in \mathcal{T}} E(\mathcal{F}^\circ) \;=\; E(\mathcal{P}^\circ)$$

and this (finally!) concludes our proof of Theorem 1.15.

Exercises 1.37 and 1.39 also answer the question why we carefully cut $\mathcal{P}$ along diagonals (as opposed to cutting it up somehow to obtain triangles): Theorem 1.15 is only true for lattice polygons. There are versions for polygons with rational and irrational coordinates but they become increasingly complicated. By cutting along diagonals we can decompose a lattice polygon into lattice segments and lattice triangles. This part becomes nontrivial already in dimension 3 and we will worry about this in Chapter 5.

In Exercise 1.41 we will look into the question as to what the coefficients of $\mathrm{ehr}_{\mathcal{P}}(n)$, for a lattice polygon $\mathcal{P}$, tell us. Let's finish this chapter by considering the constant coeffient $c_0 = \mathrm{ehr}_{\mathcal{P}}(0)$. This is the most tricky one, as we could argue that $\mathrm{ehr}_{\mathcal{P}}(0) = E(0\mathcal{P})$ and since $0\mathcal{P}$ is just a single point, we get $c_0 = 1$. This argument is flawed: we defined $\mathrm{ehr}_{\mathcal{P}}(n)$ only for $n \geq 1$. To see that this argument is, in fact, plainly wrong, let's consider

$S = \mathcal{P}_1 \cup \mathcal{P}_2 \subset \mathbb{R}^2$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are disjoint lattice polygons. Since they are disjoint, $\mathrm{ehr}_S(n) = \mathrm{ehr}_{\mathcal{P}_1}(n) + \mathrm{ehr}_{\mathcal{P}_2}(n)$. Now $0S$ is also just a point and therefore

$$1 = \mathrm{ehr}_S(0) = \mathrm{ehr}_{\mathcal{P}_1}(0) + \mathrm{ehr}_{\mathcal{P}_2}(0) = 2\,.$$

It turns out that $c_0 = 1$ is still correct but the justification will occupy most of Section 3.2. In Exercise 1.42, you will prove a more general version for Theorem 1.15 that dispenses of convexity.

1.5 Notes

Graph-coloring problems started as map-coloring problems, and so the fact that the chromatic polynomial is indeed a polynomial was proved for maps (in 1912 by George Birkhoff [15]) before Hassler Whitney proved it for graphs in 1932 [80]. As we mentioned, the first proof of the four-color theorem is due to Kenneth Appel and Wolfgang Haken [4, 5]. Theorem 7.18 is due to Richard Stanley [67]. We will give a proof from a geometric point of view in Section 7.3.

As we already mentioned, the approach of studying colorings of planar graphs through flows on their duals was pioneered by William Tutte [79], who also conceived the 5-flow Conjecture. As we mentioned, this conjecture becomes a theorem when “5” is replaced by “6”, due to Paul Seymour [62]; the 8-flow theorem had previously been shown by François Jaeger [41, 42]. Theorem 1.10 is due to Felix Breuer and Raman Sanyal [17]. We will give a proof in Section 7.4.

Order polynomials were introduced by Richard Stanley [66, 71] as ‘chromatic-like polynomials for posets’ (this is reflected in Corollary 1.14); Theorem 1.12 is due to him. We will study order polynomials in depth in Chapter 6.

Theorem 1.15 is essentially due to Georg Pick [52], whose famous formula is the subject of Exercise 1.41. In some sense, this formula marks the beginning of the study of integer-point enumeration in polytopes. Our phrasing of Theorem 1.15 suggests that it has an analogue in higher dimension, and we will study this analogue in Section 4.5.

Herbert Wilf [81] raised the question of characterizing which polynomials can occur as chromatic polynomials of graphs. This question has spawned a lot of work in the algebraic combinatorics. For example, a recent theorem of June Huh [39] says that the absolute values of the coefficients of any chromatic polynomial form a *unimodal* sequence, that is, the sequence increases up to some point, after which it decreases. Huh’s theorem had been conjectured by Ronald Read [56] almost 50 years earlier. In fact, Huh proved much more. In Chapter 7 we will study arrangements of hyperplanes and their associated characteristic polynomials. Huh and later Huh and Katz [40] proved that, up to sign, the coefficients of characteristic polynomials of hyperplane arrangements (defined over any field) form a *log-concave* sequence. We will later see the relation between chromatic and characteristic polynomials.

We finish by mentioning one general problem, which fits the "big picture" point-of-view of this introductory chapter: classify chromatic, flow, order, and Ehrhart polynomials. What we mean is: give conditions on real numbers $a_0, a_1, \dots, a_d$ that allow us to detect whether or not a given polynomial $a_d\, n^d + a_{d-1}\, n^{d-1} + \cdots + a_0$ is, say, a chromatic polynomial.

Exercises

1.18. Let $f(t) = a_d t^d + a_{d-1} t^{d-1} + \cdots + a_0 \in \mathbb{R}[t]$ be a polynomial such that $f(n)$ is an integer for every integer $n > 0$. Give a proof or a counterexample for the following statements.

(a) All coefficients a_i are integers.

(b) $f(n)$ is an integer for all $n \in \mathbb{Z}$.

(c) If $(-1)^k f(-n) \geq 0$ for all $n > 0$, then $k = \deg(f)$.

1.19. Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a bijection $\phi : V_1 \to V_2$ such that for all $u, v \in V_1$

$$uv \in E_1 \quad \text{if and only if} \quad \phi(u)\,\phi(v) \in E_2 .$$

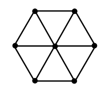
Let G be a planar graph and let G_1 and G_2 be the dual graphs for two distinct planar embeddings of G. Is it true that G_1 and G_2 are isomorphic?

If not, can you give a sufficient condition on G such that the above claim is true? (A precise characterization is rather difficult, but for a sufficient condition you might want to contemplate Steinitz's theorem [84, Ch. 4].)

1.20. Find two simple non-isomorphic graphs G and H with $\chi_G(n) = \chi_H(n)$. Can you find many (polynomial, exponential) such examples in the number of nodes? Can you make your examples arbitrarily high connected?

1.21. Find the chromatic polynomials of

(a) the wheel with d spokes (and $d+1$ nodes); for example, the wheel with 6 spokes is this:



(b) the cycle on d nodes;

(c) the path on d nodes.

1.22. Show that if G has c connected components, then n^c divides the polynomial $\chi_G(n)$.

1.23. Complete the proof of Corollary 1.2: Let G be a loopless nonempty graph on d nodes and $\chi_G(n) = c_d n^d + c_{d-1} n^{d-1} + \cdots + c_0$ its chromatic polynomial. Then

(a) the leading coefficient $c_d = 1$;

(b) the constant coefficient $c_0 = 0$;

(c) $(-1)^d \chi_G(-n) > 0$.

1.24. Prove that every **complete graph** K_d (a graph with d nodes and all possible edges between them) has exactly $d!$ acyclic orientations.

1.25. Find and prove a deletion–contraction formula for the number of acyclic orientations of a given graph.

1.26. In this exercise you will give a deletion–contraction proof of Theorem 1.5.

(a) Verify that the deletion–contraction relation (1.1) implies for $\overline{\chi}_G(n) := (-1)^n \chi_G(-n)$ that

$$\overline{\chi}_G(n) \;=\; \overline{\chi}_{G\setminus e}(n) \;+\; \overline{\chi}_{G/e}(n).$$

(b) Define $\mathcal{X}_G(n)$ as the number of compatible pairs acyclic orientation ρ and n-coloring c. Show $\mathcal{X}_G(n)$ satisfies the same deletion–contraction relation above.

(c) Infer that $\overline{\chi}_G(n) = \mathcal{X}_G(n)$ by induction on $|E|$.

1.27. The **complete bipartite graph** $K_{r,s}$ is the graph on the vertex set $V = \{1,2,\ldots,r,1',2',\ldots,s'\}$ and edges $E = \{ij' \,:\, 1 \le i \le r, 1 \le j \le s\}$. Determine

the chromatic polynomial $\chi_{K_{r,s}}(n)$ for $m,k \geq 1$.
(*Hint:* Proper n-colorings of $K_{r,s}$ correspond to pairs (f,g) of maps $f:[r] \to [n]$ and $g:[s] \to [n]$ with disjoint ranges.)

1.28. Prove Proposition 1.6: The flow-counting function $\varphi_G(n)$ is independent on the orientation of G.

1.29. Let G be a connected planar graph with dual G^*. By reversing the steps in our proof before Proposition 1.7, show that every (nowhere-zero) $\mathbb{Z}_n$-flow f on G^* naturally gives rise to n different (proper) n-colorings on G.

1.30. Prove Proposition 1.9: If G is a bridgeless connected graph, then $\varphi_G(n)$ agrees with a monic polynomial of degree $|E|-|V|+1$ with integer coefficients.

1.31. Let $G=(V,E)$ be a graph, and let n be a positive integer.

(a) Prove that

$$\varphi_G(n) \neq 0 \qquad \text{implies} \qquad \varphi_G(n+1) \neq 0\,.$$

(*Hint:* use Tutte's Lemma [78].)

(b) Even stronger, prove that

$$\varphi_G(n) \leq \varphi_G(n+1)\,.$$

(This is nontrivial. But you will easily prove this after having read Chapter 7.)

1.32. Let ${}_\rho G=(V,E,\rho)$ be an oriented graph and $n \geq 2$.

(a) Let $f:E \to \mathbb{Z}_n$ be a nowhere-zero $\mathbb{Z}_n$-flow and let $e \in E$. Show that

$$f: E \setminus \{e\} \to \mathbb{Z}_n$$

is a nowhere-zero $\mathbb{Z}_n$-flow on the contraction ${}_\rho G/e$.

(b) For $S \subseteq V$ let $E^{\text{in}}(S)$ be the *in-coming* edges, i.e., $u \to v$ with $v \in S$ and $u \in V \setminus S$ and $E^{\text{out}}(S)$ the *out-going* edges. Show that $f:E \to \mathbb{Z}_n$ is a nowhere-zero $\mathbb{Z}_n$-flow if and only if

$$\sum_{e \in E^{\text{in}}(S)} f(e) \;=\; \sum_{e \in E^{\text{out}}(S)} f(e)$$

for all $S \subseteq V$. (*Hint:* For the sufficiency contract all edges in S and $V \setminus S$.)

(c) Infer that $\varphi_G \equiv 0$ if G has a bridge.

1.33. Discover the notion of tensions.

1.34. Consider the **Petersen graph** G pictured in Figure 1.12.

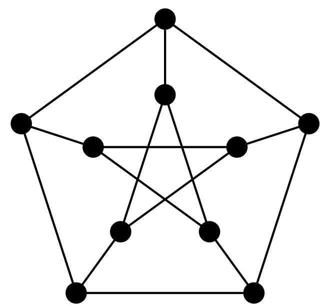


Fig. 1.12 The Petersen graph.

(a) Show that $\varphi_G(4) = 0$.

(b) Show that the polynomial $\varphi_G(n)$ has nonreal roots.

(c) Construct a planar[4] graph whose flow polynomial has nonreal roots. (*Hint:* think of the dual coloring question.)

1.35. Prove Proposition 1.13: Let ${}_\rho G = (V, E, \rho)$ be an acyclic graph and $\Pi = \Pi({}_\rho G)$ the induced poset. A map $c : V \to [n]$ is strictly compatible with the orientation ρ of G if and only if c is a strictly order preserving map $\Pi \to [n]$.

1.36. Show that $\Omega_\Pi(n)$ is a polynomial in n.

1.37. Let $\mathcal{S} = \operatorname{conv}\left\{\binom{a_1}{b_1}, \binom{a_2}{b_2}\right\}$ with $a_1, a_2, b_1, b_2 \in \mathbb{Z}$, a **lattice segment**. Show that

$$\operatorname{ehr}_{\mathcal{S}}(n) = L\,n + 1$$

where $L = \gcd(a_2 - a_1, b_2 - b_1)$, the **lattice length** of $\mathcal{S}$. Conclude further that $-\operatorname{ehr}_{\mathcal{S}}(-n)$ is exactly the number of lattice points of $n\mathcal{S}$ other than the endpoints, in other words,

$$(-1)^{\dim \mathcal{S}} \operatorname{ehr}_{\mathcal{S}}(-n) = \operatorname{ehr}_{\mathcal{S}^\circ}(n).$$

[4] The Petersen graph is a (famous) example of a nonplanar graph.

Can you find an explicit formula for $\mathrm{ehr}_{\mathcal{S}}(n)$ when $\mathcal{S}$ is a segment with *rational* endpoints?

1.38. Let $\mathcal{O}$ be a closed polygonal lattice path, i.e., the union of lattice segments, such that any vertex on $\mathcal{O}$ lies on precisely two such segments, and that topologically $\mathcal{O}$ is a closed curve. Show that

$$\mathrm{ehr}_{\mathcal{O}}(n) \;=\; L\,n$$

where L is the sum of the lattice lengths of the lattice segments that make up $\mathcal{O}$ or, equivalently, the number of lattice points on $\mathcal{O}$.

1.39. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{Z}^2$, and let $\mathcal{Q}$ be the half-open parallelogram

$$\mathcal{Q} := \{\lambda \mathbf{v}_1 + \mu \mathbf{v}_2 : 0 \le \lambda, \mu < 1\}$$

Show (for example, by tiling the plane by translates of $\mathcal{Q}$) that

$$\mathrm{ehr}_{\mathcal{Q}}(n) \;=\; A\,n^2$$

where $A = \left| \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} \right|$.

1.40. A lattice triangle $\mathrm{conv}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is **unimodular** if $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$ form a lattice basis of $\mathbb{Z}^2$.

(a) Prove that a lattice triangle is unimodular if and only if it has area $\frac{1}{2}$.

(b) Conclude that for any two unimodular triangles Δ_1 and Δ_2, there exist $T \in \mathrm{GL}_2(\mathbb{Z})$ and $\mathbf{x} \in \mathbb{Z}^2$ such that $\Delta_2 = T(\Delta_1) + \mathbf{x}$.

(c) Compute the Ehrhart polynomials of all unimodular triangles.

(d) Show that any lattice polygon can be triangulated into unimodular triangles.

(e) Use the above facts to give an alternative proof of Theorem 1.15.

1.41. Let $\mathcal{P} \subset \mathbb{R}^2$ be a lattice polygon, denote the area of $\mathcal{P}$ by A, the number of integer points inside the polygon $\mathcal{P}$ by I, and the number of integer points on the boundary of $\mathcal{P}$ by B. Prove that

$$A = I + \tfrac{1}{2}B - 1$$

(a famous formula due to Georg Alexander Pick). Deduce from this formulas for the coefficients of the Ehrhart polynomial of $\mathcal{P}$.

1.42. Let $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^2$ be lattice polygons, such that $\mathcal{Q}$ is contained in the interior of $\mathcal{P}$. Generalize Exercise 1.41 (i.e., both a version of Pick's theorem and the accompanying Ehrhart polynomial) to the "polygon with a hole" $\mathcal{P} - \mathcal{Q}$. Generalize your formulas to a lattice polygon with n "holes" (instead of one).

Chapter 2
Partially Ordered Sets

The mathematical phenomenon always develops out of simple arithmetic, so useful in everyday life, out of numbers, those weapons of the gods: the gods are there, behind the wall, at play with numbers.
Le Corbusier

Partially ordered sets, *posets* for short, made an appearance twice so far. First (in Section 1.3) as a class of interesting combinatorial objects with a rich counting theory that is intimately related to colorings and, second (in Section 1.4), as a natural bookkeeping structure for geometric subdivisions of polygons. In particular, the stage for the principle of overcounting-and-correcting, more commonly referred to as inclusion–exclusion, is naturally set in the theory of posets. Our agenda in this chapter is twofold: we need to introduce machinery that will be crucial tools in later chapters, but we will also prove our first combinatorial reciprocity theorems in a general setting, from first principles—that is, without the geometric intuition that will appear in later chapters (where we will revisit the theorems from this chapter in light of the geometry). Let's recall that for us, a poset Π is a finite set with a binary relation $\preceq_\Pi$ that is reflexive, transitive, and anti-symmetric.

2.1 Order Ideals and the Incidence Algebra

Let's go back to the problem of counting order preserving maps, i.e., maps $\phi : \Pi \to [n]$ that satisfy

$$a \ \preceq_\Pi \ b \quad \Longrightarrow \quad \phi(a) \ \le \ \phi(b)$$

for all $a,b \in \Pi$. The preimages $\phi^{-1}(j)$, for $j = 1,2,\ldots,n$, partition Π and uniquely identify ϕ, but from a poset point of view they do not have enough structure. A better perspective comes from the following observation: Let $\phi : \Pi \to [2]$ be an order preserving map into the 2-chain, and let $I := \phi^{-1}(1)$. Now

$$y \in I \text{ and } x \preceq_\Pi y \quad \Longrightarrow \quad x \in I .$$

A subset $I \subseteq \Pi$ with this property is called an **order ideal** of Π. Conversely, if $I \subseteq \Pi$ is an order ideal, then $\phi : \Pi \to [2]$ with $\phi^{-1}(1) = I$ defines an order preserving map. Thus, order preserving maps $\phi : \Pi \to [2]$ are in bijection with the order ideals of Π, of which there are exactly $\Omega_\Pi(2)$ many. Although we will not need them here, we remark that, dually, the complement $F = \Pi \setminus I$ of an order ideal is characterized by the property that $x \succeq y \in F$ implies $x \in F$. Such a set is called a **dual order ideal** or **filter**.

To characterize general order preserving maps into chains in terms of Π, we note that every order ideal of $[n]$ is **principal**, that is, every order ideal $I \subseteq [n]$ is of the form

$$I = \{j \in [n] : j \le k\} = [k]$$

for some k. In particular, the preimage $\phi^{-1}([k])$ of an order ideal $[k] \subseteq [n]$ is an order ideal of Π, and this gives us the following bijection.

Proposition 2.1. *Order preserving maps $\phi : \Pi \to [n]$ are in bijection with multichains*[1] *of order ideals*

$$\varnothing \;=\; I_0 \;\subseteq\; I_1 \;\subseteq\; I_2 \;\subseteq\; \cdots \;\subseteq\; I_n \;=\; \Pi$$

of length n. The map ϕ is strictly order preserving if and only if $I_j \setminus I_{j-1}$ is an antichain for all $j = 1,2,\ldots,n$.

Proof. We only need to argue the second part. We observe that ϕ is strictly order preserving if and only if there are no elements $x \prec y$ with $\phi(x) = \phi(y)$. Hence, ϕ is strictly order preserving if and only if $\phi^{-1}(j) = I_j \setminus I_{j-1}$ does not contain a pair of comparable elements. ☐

The collection $\mathcal{J}(\Pi)$ of order ideals of Π is itself a poset under set inclusion, which we call the **lattice of order ideals** or the **Birkhoff lattice**[2] of Π. What we just showed is that $\Omega_\Pi(n)$ counts the number of multichains in $\mathcal{J}(\Pi) \setminus \{\varnothing,\Pi\}$. The next problem we address is counting multichains of length n in general posets. To that end, we introduce an algebraic gadget: The **incidence algebra** $I(\Pi)$ is a $\mathbb{C}$-vector space spanned by those functions $\alpha : \Pi \times \Pi \to \mathbb{C}$ that satisfy

[1] A **multichain** is a sequence of comparable elements, where we allow repetition.

[2] The reason for this terminology will become clear shortly.

$$\alpha(x,y) \;=\; 0 \quad \text{whenever} \quad x \not\preceq y\,.$$

We define the product of $\alpha,\beta : \Pi \times \Pi \to \mathbb{C}$ as

$$(\alpha * \beta)(r,t) \;:=\; \sum_{r \preceq s \preceq t} \alpha(r,s)\beta(s,t)\,,$$

and together with $\delta \in I(\Pi)$ defined by

$$\delta(x,y) \;:=\; \begin{cases} 0 & \text{if } x \neq y, \\ 1 & \text{if } x = y, \end{cases}$$

this gives $I(\Pi)$ the structure of an associative $\mathbb{C}$-algebra with unit δ. (Those readers for whom this is starting to feel like linear algebra are on the right track.) As we will see below, a distinguished role is played by the **zeta function** $\zeta \in I(\Pi)$ defined by

$$\zeta(x,y) \;:=\; \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

For the time being, the power of zeta functions lie in their powers.

Proposition 2.2. *Let Π be a finite poset and $x,y \in \Pi$. Then $\zeta^n(x,y)$ equals the number of multichains*

$$x \;=\; x_0 \;\preceq\; x_1 \;\preceq\; \cdots \;\preceq\; x_n = y$$

of length n.

Proof. For $n = 1$, we have that $\zeta(x,y) = 1$ if and only if $x = x_0 \preceq x_1 = y$. Arguing by induction, we can assume that $\zeta^{n-1}(x,y)$ is the number of multichains of length $n-1$ for all $x,y \in \Pi$, and we calculate

$$\zeta^n(x,z) \;=\; (\zeta^{n-1} * \zeta)(x,z) \;=\; \sum_{x \preceq y \preceq z} \zeta^{n-1}(x,y)\,\zeta(y,z)\,.$$

Every summand on the right is the number of multichains of length $n-1$ ending in y that can be extended to z. $\square$

As an example, the zeta function for the poset in Figure 2.1 is given in matrix form as

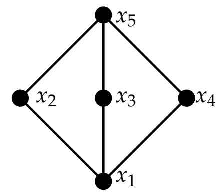


Fig. 2.1 A sample poset.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We encourage the reader to see Proposition 2.2 in action and to compute powers of this matrix.

As a milestone, Proposition 2.2 implies the following presentation of the order polynomial of Π which we introduced in Section 1.3.

Corollary 2.3. *For a finite poset Π, let $\mathcal{J} = \mathcal{J}(\Pi)$ be its lattice of order ideals and ζ the zeta function of $\mathcal{J}$. The order polynomial associated with Π is given by*

$$\Omega_\Pi(n) \;=\; \zeta^n(\varnothing,\Pi)\,.$$

Identifying $\Omega_\Pi(n)$ with the evaluation of a power of ζ does not suggest that $\Omega_\Pi(n)$ is the restriction of a polynomial (which we know to be true from Exercise 1.36) but this impression is misleading: Let $\eta \in I(\Pi)$ be defined by

$$\eta(x,y) \;:=\; \begin{cases} 1 & \text{if } x \prec y, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\zeta = \delta + \eta$ and hence

$$\zeta^n(x,y) \;=\; (\delta+\eta)^n(x,y) \;=\; \sum_{k=0}^{n} \binom{n}{k} \eta^k(x,y)\,. \tag{2.1}$$

Exercise 2.14 asserts that the sum on the right stops at the index $k = |\Pi|$ and is thus a polynomial in n of degree $|\Pi|$.

The arguments in the preceding paragraph are not restricted to posets formed by order ideals, but hold more generally for any poset Π that has a **minimum** $\hat{0}$ and a **maximum** $\hat{1}$, i.e., $\hat{0}$ and $\hat{1}$ are elements in Π that satisfy

$\hat{0} \preceq x \preceq \hat{1}$ for all $x \in \Pi$. (For example, the Birkhoff lattice $\mathcal{J}(\Pi)$ has minimum $\varnothing$ and maximum Π.) Let's record this:

Proposition 2.4. *Let Π be a finite poset with minimum $\hat{0}$, maximum $\hat{1}$, and zeta function ζ. Then $\zeta^n(\hat{0},\hat{1})$ is a polynomial in n.*

To establish a reciprocity theorem for $\Omega_\Pi(n)$, we'd like to evaluate $\zeta^n(\varnothing,\Pi)$ at negative integers n, so we first need to understand when an element $\alpha \in I(\Pi)$ is invertible. To this end, let's pause and make the incidence algebra a bit more tangible.

Choose a **linear extension** of Π, that is, we label the $d = |\Pi|$ elements of Π by $p_1, p_2, \dots, p_d$ such that $p_i \preceq p_j$ implies $i \le j$. (That such a labeling exists is the content of Exercise 2.15.) This allows us to identify $I(\Pi)$ with a subalgebra of the upper triangular $(d \times d)$-matrices by setting

$$\alpha = \big(\alpha(p_i, p_j)\big)_{1 \le i,j \le d} \,.$$

For example, for the poset D_{10} given in Figure 1.8, a linear extension is given by $(p_1, p_2, \dots, p_{10}) = (1,5,2,3,7,10,4,6,9,8)$ and the incidence algebra consists of matrices of the form

$$\begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & 5 & 2 & 3 & 7 & 10 & 4 & 6 & 9 & 8 \end{array} \\
\begin{array}{c} 1 \\ 5 \\ 2 \\ 3 \\ 7 \\ 10 \\ 4 \\ 6 \\ 9 \\ 8 \end{array} &
\begin{pmatrix}
* & * & * & * & * & * & * & * & * & * \\
 & * & & & & * & & & & \\
 & & * & & & * & * & * & & * \\
 & & & * & & & & * & * & \\
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 & & & & & & * & & & * \\
 & & & & & & & * & & \\
 & & & & & & & & * & \\
 & & & & & & & & & *
\end{pmatrix}
\end{array}$$

where the stars are the possible non-zero entries for the elements in $I(\Pi)$. This linear-algebra perspective affords a simple criterion for when α is invertible.

Proposition 2.5. *A transformation $\alpha \in I(\Pi)$ is invertible if and only if $\alpha(x,x) \neq 0$ for all $x \in \Pi$.*

2.2 The Möbius Function and Order Polynomial Reciprocity

We now return to the stage set up by Corollary 2.3, namely, that

$$\Omega_\Pi(n) \;=\; \zeta^n_{\mathcal{J}(\Pi)}(\varnothing,\Pi)\,.$$

We'd like to use this setup to compute $\Omega_\Pi(-n)$; thus we need to invert the zeta function of Π. Such an inverse exists by Proposition 2.5, and we call $\mu := \zeta^{-1}$ the **Möbius function** of Π. For example, the Möbius function of the poset in Figure 2.1 is given in matrix form as

$$\begin{pmatrix} 1 & -1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is apparent that one can compute the Möbius function recursively, and in fact, unravelling the condition that $(\mu * \zeta)(x,z) = \delta(x,z)$ for all $x,z \in \Pi$ gives

$$\begin{aligned} \mu(x,z) \;&=\; -\sum_{x \prec y \preceq z} \mu(x,y) \;=\; -\sum_{x \preceq y \prec z} \mu(y,z) \quad \text{for } x \prec z \text{ and} \\ \mu(x,x) \;&=\; 1\,. \end{aligned} \tag{2.2}$$

As a notational remark, the functions ζ, δ, μ, and η depend on the underlying poset Π, so we will sometimes write ζ_Π, δ_Π, etc., to make this dependence clear. For an example, let's consider the Möbius function of the **Boolean lattice** B_d, the partially ordered set of all subsets of $[d]$ ordered by inclusion. For two subsets $S \subseteq T \subseteq [d]$, we have that $\mu_{B_d}(S,T) = 1$ whenever $S = T$ and $\mu_{B_r}(S,T) = -1$ whenever $|T \setminus S| = 1$. Although this is little data, we venture that

$$\mu_{B_d}(S,T) = (-1)^{|T \setminus S|}. \tag{2.3}$$

We dare the reader to prove this from first principles, or to appeal to the results in Exercise 2.21 after realizing that B_d is the d-fold product of a 2-chain.

Towards the combinatorial reciprocity theorem for order polynomials (Theorem 1.12) we note the following.

Proposition 2.6.

$$\Omega_\Pi(-n) \;=\; \zeta^{-n}_{\mathcal{J}}(\varnothing,\Pi) \;=\; \mu^n_{\mathcal{J}}(\varnothing,\Pi)\,, \tag{2.4}$$

where $\mathcal{J} = \mathcal{J}(\Pi)$ is the Birkhoff lattice of Π.

This proposition is strongly suggested by our notation but nevertheless requires a proof.

Proof. Recall that $\zeta = \delta + \eta$. Hence

$$\zeta^{-1} \;=\; (\delta+\eta)^{-1} \;=\; \delta - \eta + \eta^2 - \cdots + (-1)^d \eta^d,$$

by the geometric series and Exercise 2.14. If we now take powers of ζ^{-1} and again appeal to the nilpotency of η, we calculate

$$\zeta^{-n} \;=\; \sum_{k=0}^{d} (-1)^k \binom{n+k-1}{k} \eta^k.$$

Thus the expression of ζ^n as a polynomial given in (2.1) together with the fundamental combinatorial reciprocity for binomial coefficients (0.2) given in the very beginning of the book proves the claim. $\square$

The right-hand side of (2.4) is

$$\mu_{\mathcal{J}}^n(\varnothing,\Pi) \;=\; \sum \mu_{\mathcal{J}}(I_0,I_1)\,\mu_{\mathcal{J}}(I_1,I_2)\cdots\mu_{\mathcal{J}}(I_{n-1},I_n) \tag{2.5}$$

where the sum is over all multichains of order ideals

$$\varnothing \;=\; I_0 \;\subseteq\; I_1 \;\subseteq\; \cdots \;\subseteq\; I_n \;=\; \Pi$$

of length n. Our next goal is thus to understand the evaluation $\mu_{\mathcal{J}}(K,M)$ where $K \subseteq M \subseteq \Pi$ are order ideals. This evaluation depends only on

$$[K,M] := \{L \in \mathcal{J} : K \subseteq L \subseteq M\},$$

the **interval** from K to M in the Birkhoff lattice $\mathcal{J}$.

Theorem 2.7. *Let Π be a finite poset and $K \subseteq M$ order ideals in $\mathcal{J} = \mathcal{J}(\Pi)$. Then*

$$\mu_{\mathcal{J}}(K,M) \;=\; \begin{cases} (-1)^{|M\setminus K|} & \text{if } M \setminus K \text{ is an antichain,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let's first consider the (easier) case that $M \setminus K$ is an antichain. In this case $K \cup A$ is an order ideal for all $A \subseteq M \setminus K$. In other words, the interval $[K,M]$ is isomorphic[3] to the Boolean lattice B_r for $r = |M \setminus K|$ and hence, with (2.3), we conclude $\mu_{\mathcal{J}}(K,M) = (-1)^r$.

[3] Two posets Π and Θ are **isomorphic** if there is a bijection $f : \Pi \to \Theta$ that satisfies $x \preceq_\Pi y \iff f(x) \preceq_\Theta f(y)$.

The case that $M \setminus K$ contains comparable elements is a bit more tricky. We can argue by induction on the length of the interval $[K,M]$. The base case is given by the situation that $M \setminus K$ consists of exactly two comparable elements $m \prec M$ and hence

$$\mu_{\mathcal{J}}(K,M) \;=\; -\;\mu_{\mathcal{J}}(K,K) \;-\; \mu_{\mathcal{J}}(K,K\cup\{m\}) \;=\; -1 \;-\; (-1) \;=\; 0$$

since $K \cup \{m\}$ is an order ideal that covers K in $\mathcal{J}$.

For the induction step, we use (2.2), i.e.,

$$\mu_{\mathcal{J}}(K,M) \;=\; -\sum \mu_{\mathcal{J}}(K,L)$$

where the sum is over all order ideals L such that $K \subseteq L \subset M$. By induction hypothesis, $\mu_{\mathcal{J}}(K,L)$ is zero unless $L \setminus K$ is an antichain and thus

$$\mu_{\mathcal{J}}(K,M) \;=\; -\sum \left\{ (-1)^{|L\setminus K|} : \begin{array}{l} K \subseteq L \subset M \text{ order ideal,} \\ L \setminus K \text{ is an antichain} \end{array} \right\},$$

where we have used the already-proven part of the theorem. Now let $m \in M \setminus K$ be a minimal element. The order ideals L in the above sum can be partitioned into those containing m and those who don't. Both parts of this partition have the same size: if $m \notin L$, then $L \cup \{m\}$ is also an order ideal. If $m \in L$, then $L \setminus \{m\}$ is an admissible order ideal as well. (You should check this.) Hence, the positive and negative terms cancel each other and $\mu_{\mathcal{J}}(K,M) = 0$. $\square$

With this we can give a purely combinatorial proof of Theorem 1.12.

Proof of Theorem 1.12. With Theorem 2.7, the right-hand side of (2.5) becomes $(-1)^{|\Pi|}$ times the number of multichains $\varnothing = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = \Pi$ of order ideals such that $I_j \setminus I_{j-1}$ is an antichain for $j = 1,\ldots,n$. By Proposition 2.1 this is exactly $(-1)^{|\Pi|}\Omega^{\circ}_{\Pi}(n)$, and this proves Theorem 1.12. $\square$

Our proof also gives us some structural insights into $\Omega_{\Pi}(n)$.

Corollary 2.8. *Let Π be a poset and $1 \le m \le |\Pi|$. Then $\Omega(-k) = 0$ for all $0 < k < m$ if and only if Π contains an m-chain.*

2.3 Zeta Polynomials, Distributive Lattices, and Eulerian Posets

We now take a breath and see how far we can generalize (the assumptions in) Theorem 1.12. Our starting point is Proposition 2.4: for a poset Π that has a minimum $\hat{0}$ and maximum $\hat{1}$, the evaluation

$$Z_\Pi(n) := \zeta^n(\hat{0},\hat{1})$$

is a polynomial in n, the **zeta polynomial** of Π. For example, if we augment the poset D_{10} in Figure 1.8 by a maximal element (think of the number 0, which is divisible by all positive integers), Exercise 2.17 gives the accompanying zeta polynomial as

$$Z_\Pi(n) = \tfrac{1}{24}n^4 + \tfrac{11}{12}n^3 + \tfrac{35}{24}n^2 - \tfrac{17}{12}n + 1\,. \tag{2.6}$$

In analogy with the combinatorial reciprocity theorem for order polynomials (Theorem 1.12)—which are, after all, zeta polynomials of posets formed by order ideals—we now seek interpretations for evaluations of zeta polynomials at negative integers. Analogous to (2.4), we have

$$Z_\Pi(-n) = \zeta^{-n}(\hat{0},\hat{1}) = \mu^n(\hat{0},\hat{1})$$

where μ is the Möbius function of Π. Our above example zeta function (2.6) illustrates that the quest for interpretations at negative evaluations is nontrivial: here we compute

$$Z_\Pi(-2) = 3 \qquad \text{and} \qquad Z_\Pi(-3) = -3$$

and so any hope of a simple counting interpretation of $Z_\Pi(-n)$ or $-Z_\Pi(-n)$ is shattered. On a more optimistic note, we can repeat the argument behind (2.5):

$$Z_\Pi(-n) = \mu^n(\hat{0},\hat{1}) = \sum \mu(x_0,x_1)\,\mu(x_1,x_2)\cdots\mu(x_{n-1},x_n) \tag{2.7}$$

where the sum is over all multichains

$$\hat{0} = x_0 \preceq x_1 \preceq \cdots \preceq x_n = \hat{1}$$

of length n. The key property that put (2.5) to work in our proof of Theorem 2.7 (and subsequently, our proof of Theorem 1.12) was that every summand on the right-hand side of (2.5) was either 0 or (the same) constant. We thus seek a class of posets where a similar property holds in (2.7).

For two elements x and y in a poset Π, consider all least upper bounds of x and y, i.e., all $z \in \Pi$ such that $x \preceq z$ and $y \preceq z$ and there is no $w \prec z$ with the same property. If such a least upper bound of x and y exists and is unique, we call it the **join** of x and y and denote it by $x \vee y$. Dually, consider all greatest lower bounds of x and y; if a greatest lower bound exits and is unique, we call it the **meet** of x and y and denote it by $x \wedge y$.

A **lattice**[4] is a poset in which meets and joins exist for any pair of elements. Note that any finite lattice will necessarily have a minimum $\hat{0}$ and a maximum element $\hat{1}$. A lattice Π is **distributive** if meets and joins satisfy the distributive laws

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z) \qquad \text{and} \qquad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$$

for all $x, y, z \in \Pi$. The reason we are interested in distributive lattices is the following famous result, whose proof is subject to Exercise 2.18.

Theorem 2.9. *Every finite distributive lattice is isomorphic to the poset of order ideals of some poset.*

Let's experience the consequences of this theorem. Given a finite distributive lattice Π, we now know that the Möbius-function values on the right-hand side of (2.7) can be interpreted as stemming from the poset of order ideals of some other poset. But this means that we can apply Theorem 2.7, in precisely the same way we applied Theorem 2.7 in our proof of Theorem 1.12: the right-hand side of (2.7) becomes $(-1)^{|\Pi|}$ times the number of multichains $\hat{0} = x_0 \preceq x_1 \preceq \cdots \preceq x_n = \hat{1}$ such that the corresponding differences of order ideals are all antichains. A moment's thought reveals that this last condition is equivalent to the fact that each interval $[x_j, x_{j+1}]$ is a Boolean lattice. What we have just proved is a combinatorial reciprocity theorem which, in a sense, generalizes that of order polynomials (Theorem 1.12):

Theorem 2.10. *Let Π be a finite distributive lattice. Then $(-1)^{|\Pi|} Z_\Pi(-n)$ equals the number of multichains $\hat{0} = x_0 \preceq x_1 \preceq \cdots \preceq x_n = \hat{1}$ such that each interval $[x_j, x_{j+1}]$ is a Boolean lattice.*

There is another class of posets that comes with a combinatorial reciprocity theorem stemming from (2.7). To introduce it, we need a few more definitions. A finite poset Π is **graded** if every maximal chain in Π has the same length, which we call the **rank** of Π. The **length** $l[x, y]$ of an interval $[x, y]$ in

[4] This *lattice* is not to be confused with the integer lattice $\mathbb{Z}^2$ that made an appearance in Section 1.4 and whose higher-dimensional cousins will play a central role in later chapters. Both meanings of *lattice* are well furnished in the mathematical literature; we hope that they will not be confused in this book.

a poset Π is the length of a maximal chain in $[x,y]$. A graded poset that has a minimal and a maximal element is **Eulerian** if its Möbius function is

$$\mu(x,y) = (-1)^{l[x,y]}.$$

We have seen examples of Eulerian posets earlier, for instance, Boolean lattices. Another important class of Eulerian posets are formed by faces of polyhedra, which we will study in the next chapter.

What happens with (2.7) when the underlying poset Π is Eulerian? In this case, the Möbius-function values on the right-hand side are determined by the interval length, and so each summand on the right is simply $(-1)^r$ where r is the rank of Π. But then (2.7) says that $Z_\Pi(-n)$ equals $(-1)^r$ times the number of multichains of length n, which is $\zeta^n(\hat{0},\hat{1}) = Z_\Pi(n)$. This argument yields a reciprocity theorem that relates the zeta polynomial of Π to itself:

Theorem 2.11. *Let Π be an Eulerian poset of rank r. Then*

$$Z_\Pi(-n) = (-1)^r Z_\Pi(n)\,.$$

We will come back to this result in connection with the combinatorial structure of polytopes.

2.4 Möbius Inversion

Our use of the Möbius function to prove Theorem 1.12 is by far not the only instance where this function is useful—zeta and Möbius functions make a prominent appearance in practically all counting problems in which posets play a structural role. The setting is the following: For a fixed poset Π, we want to know a certain function $f : \Pi \to \mathbb{C}$ but all we know is the function

$$g(y) \;=\; \sum_{x \preceq y} f(x)\, c(x,y) \tag{2.8}$$

where we can think of the $c(x,y)$ as some coefficients distorting the given function f. In such a situation, can we infer f from g? The typical situation is given by

$$f_{\preceq}(y) \;:=\; \sum_{x \preceq y} f(x).$$

The right (algebraic) setting in which to address such questions is the incidence algebra: Every $\alpha \in I(\Pi)$ defines a linear transformation on $\mathbb{C}^\Pi$ via

$$(\alpha f)(y) := \sum_{x \in \Pi} f(x)\,\alpha(x,y). \tag{2.9}$$

In the warm-up Exercise 2.13 you are asked to verify that, with the above definition, $I(\Pi)$ yields a right action on $\mathbb{C}^{\Pi}$. Thus, in (2.8), f can be recovered from g whenever c is invertible. For the zeta function, the procedure of recovering f from $f_{\preceq}$ goes by the name of Möbius inversion.

Theorem 2.12 (Möbius inversion). *Let Π be a poset, μ its associated Möbius function, and $f : \Pi \to \mathbb{C}$. Then*

$$f_{\preceq}(y) = \sum_{x \preceq y} f(x) \qquad \Longleftrightarrow \qquad f(y) = \sum_{x \preceq y} f_{\preceq}(x)\,\mu(x,y).$$

and, likewise,

$$f_{\succeq}(y) = \sum_{x \succeq y} f(x) \qquad \Longleftrightarrow \qquad f(y) = \sum_{x \succeq y} \mu(y,x)\,f_{\succeq}(x).$$

Proof. The statement of the theorem is simply

$$\mu f_{\preceq} = (\mu * \zeta) f = f$$

where we use $\mu = \zeta^{-1}$. However, it is instructive to do the yoga of Möbius inversion

$$\begin{aligned}
(\mu f_{\preceq})(z) &= \sum_{y \preceq z} f_{\preceq}(y)\,\mu(y,z) = \sum_{y \preceq z} \sum_{x \preceq y} f(x)\,\mu(y,z) \\
&= \sum_{x \preceq z} f(x) \sum_{x \preceq y \preceq z} \mu(y,z) = \sum_{x \preceq z} f(x)\,\delta(x,z) \\
&= f(z).
\end{aligned}$$

The second statement is verified the same way. □

In a nutshell this is what we implicitly used in our treatment of Ehrhart theory for lattice polygons in Section 1.4. The subdivision of a lattice polygon $\mathcal{P}$ into triangles, edges, and vertices is a genuine poset Π under inclusion. The function $f(x)$ is the number of lattice points in $x \subseteq \mathbb{R}^2$ and we were interested in the evaluation of $f(\mathcal{P}) = (\mu f_{\subseteq})(\mathcal{P})$.

2.5 Notes

Posets and lattices originated in the nineteenth century but become subjects of their own rights with the work of Garrett Birkhoff, who proved Theorem 2.9 [14], and Philip Hall [34].

The oldest type of Möbius function is the one studied in number theory, which is the Möbius function (in a combinatorial sense) of the divisor lattice (see Exercise 2.22(c)). The systematic study of Möbius functions of general posets was initiated by Gian–Carlo Rota's famous paper [58] which arguably started modern combinatorics. Rota's paper also put the idea of incidence algebras on firm ground, but it can be traced back much further to Richard Dedekind and Eric Temple Bell [72, Chapter 3]. Theorem 2.7 is known as *Rota's crosscut theorem*.

As we already mentioned in Chapter 1, order polynomials were introduced by Richard Stanley [66, 71] as 'chromatic-like polynomials for posets'. Stanley's paper [71] also introduced *order polytopes*, which form the geometric face of order polynomials, as we will see in later chapters. Stanley introduced the zeta polynomial of a poset in [68], the paper that inspired the title of our book. Stanley also initiated the study of Eulerian posets in [70], though, in his own words, "they had certainly been considered earlier."

For (much) more on posets, lattices, and Möbius function, we recommend [65] and [72, Chapter 3], which contains numerous open problems; we mention one representative:

Let Π_n be the set all partitions (whose definition is given in (4.1) below) of a fixed positive integer n. We order the elements of Π_n by *refinement*, i.e., given two partitions $(a_1, a_2, \ldots, a_j)$ and $(b_1, b_2, \ldots, b_k)$ of n, we say that

$$(a_1, a_2, \ldots, a_j) \preceq (b_1, b_2, \ldots, b_k)$$

if the parts $a_1, a_2, \ldots, a_j$ can be partitioned into blocks whose sums are $b_1, b_2, \ldots, b_k$. Find the Möbius function of Π_n.

Exercises

2.13. Show that (2.9) defines a right action of $I(\Pi)$ on $\mathbb{C}^{\Pi}$. That is, $I(\Pi)$ gives rise to a vector space of linear transformations on $\mathbb{C}^{\Pi}$ and $(\alpha * \beta) f = \beta(\alpha f)$, for any $\alpha, \beta \in I(\Pi)$.

2.14. Let Π be a finite poset and recall that $\zeta = \delta + \eta$ where $\eta : \Pi \times \Pi \to \{0,1\}$ with $\eta(x,y) = 1$ whenever $x \prec y$.

(a) Show that for $x \preceq y$,

$$\eta^k(x,y) \;=\; \#\{x = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{k-1} \prec x_k = y\},$$

the number of *strict* chains of length k in the interval $[x,y]$.

(b) Infer that η is *nilpotent*, that is, $\eta^k \equiv 0$ for $k > |\Pi|$.

(c) For $x,y \in \Pi$, do you know what $(2\delta - \zeta)^{-1}(x,y)$ counts?

(d) Show that $\eta^n_{\mathcal{J}(\Pi)}(\varnothing,\Pi)$ is exactly the number of surjective order preserving maps $\Pi \to [n]$.

2.15. Show that every finite poset Π has a linear extension. (*Hint:* you can argue graphically by reading the Hasse diagram or, more formally, by induction on $|\Pi|$.)

2.16. For fixed $k,n \in \mathbb{Z}_{>0}$ consider the map $g : B_k \to \mathbb{Z}_{>0}$ given by

$$g(T) \;=\; |T|^n.$$

Show that

$$k!\,S(n,k) \;=\; (g\,\mu_{B_k})([k])$$

where $S(n,k)$ is the **Stirling number** of the second kind. (*Hint:* $k!\,S(n,k)$ counts surjective maps $[n] \to [k]$.)

2.17. Compute the zeta polynomial of the poset D_{10} (see Figure 1.8) appended by a maximal element.

2.18. Prove Theorem 2.9: Every finite distributive lattice is isomorphic to a poset of order ideals of some poset. (*Hint:* Given a distributive lattice Π, consider the subposet Θ consisting of all **join irreducible** elements, i.e., those elements that cannot be written as the join of some other elements. Show that Θ is isomorphic to $\mathcal{J}(\Pi)$.)

2.19. Let Π be a finite graded poset that has a minimal and a maximal element, and define the **rank** of $x \in \Pi$ as the length of a maximal chain ending in x. Prove that Π is Eulerian if and only if for all $x \prec y$ the interval $[x,y]$ has as many elements of even rank as of odd rank.

2.20. State and prove a result analogous to Corollary 2.8 for distributive lattices.

2.21. For posets $(\Pi_1, \preceq_1)$ and $(\Pi_2, \preceq_2)$, we define their **(direct) product** with underlying set $\Pi_1 \times \Pi_2$ and partial order

$$(x_1, x_2) \preceq (y_1, y_2) \qquad :\Longleftrightarrow \qquad x_1 \preceq_1 y_1 \text{ and } x_2 \preceq_2 y_2 .$$

(a) Show that every interval $[(x_1, x_2), (y_1, y_2)]$ of $\Pi_1 \times \Pi_2$ is of the form $[x_1, y_1] \times [x_2, y_2]$.

(b) Show that $\mu_{\Pi_1 \times \Pi_2}((x_1, x_2), (y_1, y_2)) = \mu_{\Pi_1}(x_1, y_1)\, \mu_{\Pi_2}(x_2, y_2)$.

(c) Show that the Boolean lattice B_n is isomorphic to the n-fold product of the chain $[2]$, and conclude that for $S \subseteq T \subseteq [n]$

$$\mu_{B_n}(S, T) \;=\; (-1)^{|T \setminus S|}.$$

2.22. (a) Let $\Pi = [d]$ be the d-chain. Show that for $1 \le i < j \le d$

$$\mu_{[d]}(i, j) \;=\; \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } i + 1 = j, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Write out the statement that Möbius inversion gives in this explicit case and interpret it along the lines of the Fundamental Theorem of Calculus.

(c) The Möbius function in number theory is the function $\mu : \mathbb{Z}_{>0} \to \mathbb{Z}$ defined for $n \in \mathbb{Z}_{>0}$ as $\mu(n) = 0$ if n is not *squarefree*, that is, if n is divisible by a proper prime power and otherwise as $\mu(n) = (-1)^r$ if n is the product of r distinct primes. Show that for given $n \in \mathbb{Z}_{>0}$, the partially ordered set D_n of divisors of n is isomorphic to a direct product of chains and use Exercise 2.21 to verify that $\mu(n) = \mu_{D_n}(1, n)$.

2.23. Given sets $S_1, S_2, \dots, S_n$, let Π be the poset of all possible intersections of these sets, including the "empty intersection" $S_1 \cup S_2 \cup \cdots \cup S_n$, ordered by the subset relation $\subseteq$. Compute the Möbius function of Π, and apply the Möbius inversion formula for the function $f(S) = |S|$ (you should think about the appropriate partner function g) to derive the *inclusion–exclusion principle*

$$|S_1 \cup S_2 \cup \cdots \cup S_n| = \sum_{1 \le j \le n} |S_j| \;-\; \sum_{1 \le j < k \le n} |S_j \cap S_k| \;+\; \cdots \;+\; (-1)^{n-1} |S_1 \cap S_2 \cap \cdots \cap S_n| .$$

Chapter 3

Polyhedral Geometry

One geometry cannot be more true than another; it can only be more convenient.
Jules Henri Poincaré

In this chapter we define the most convenient geometry for the combinatorial objects from Chapter 1. To give a first impression of how geometry naturally enters our combinatorial picture, let's return to the problem of counting multisubsets of size d of $[n]$. Every such multiset corresponds to a tuple $(t_1, t_2, \dots, t_d) \in \mathbb{Z}^d$ such that

$$1 \ \le \ t_1 \ \le \ t_2 \ \le \ \cdots \ \le \ t_d \ \le \ n\,.$$

Forgetting about the integrality of the t_i, we obtain a genuine geometric object as the solutions to this system of $d+1$ linear inequalities:

$$n\,\Delta_d \ = \ \left\{ \mathbf{x} \in \mathbb{R}^d \ : \ 1 \le x_1 \le x_2 \le \cdots \le x_d \le n \right\}.$$

The d-multisubsets correspond exactly to the integer lattice points $n\,\Delta_d \cap \mathbb{Z}^d$. The set $n\,\Delta_d$ is a *polyhedron*: a set defined by finitely many linear inequalities. Polyhedra constitute a rich class of geometric objects, rich enough to capture much of the enumerative combinatorics that we pursue in this book.

Besides introducing machinery to handle polyhedra, our main emphasis in this chapter is on the faces of a given polyhedron. They form a poset that is naturally graded by dimension, and counting the faces in each dimension gives rise to the famous *Euler–Poincaré formula*. This identity is at play (often behind the scenes) in practically all combinatorial reciprocity theorems that we will encounter in later chapters.

3.1 Polyhedra, Cones, and Polytopes

The building blocks for our geometric objects are given by halfspaces. To this end, we define an **affine hyperplane** as a set of the form

$$H \ := \ \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \ = \ b \right\}$$

for some normal $\mathbf{a} \in \mathbb{R}^d \setminus \{0\}$ and displacement $b \in \mathbb{R}$. We call H a **linear hyperplane** if $\mathbf{0} \in H$ or, equivalently, $b = 0$. Every hyperplane H subdivides the ambient space $\mathbb{R}^d$ into the two closed **halfspaces**

$$H^{\geq} := \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \geq b \right\} \qquad \text{and} \qquad H^{\leq} := \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \leq b \right\}. \quad (3.1)$$

From such halfspaces we can manufacture more complex objects that are bounded by hyperplanes: A **polyhedron** $\mathcal{Q} \subseteq \mathbb{R}^d$ is the intersection of finitely many halfspaces such as the one shown in Figure 3.1.

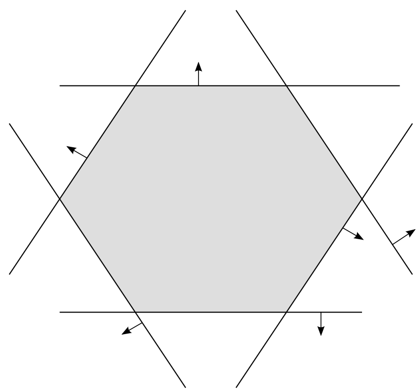

Fig. 3.1 A bounded polyhedron in the plane.

The term *polyhedron* appears in many parts of mathematics, unfortunately with different connotations. To be more careful, we just defined **convex polyhedra**. A set $S \subseteq \mathbb{R}^d$ is **convex** if for every $\mathbf{p}, \mathbf{q} \in S$, the line segment

$$[\mathbf{p}, \mathbf{q}] := \{(1-\lambda)\mathbf{p} + \lambda\mathbf{q} \ : \ 0 \leq \lambda \leq 1\}$$

with endpoints $\mathbf{p}$ and $\mathbf{q}$ is contained in S. Since hyperplanes and halfspaces are convex, a finite intersection of halfspaces produces a convex object. Nevertheless, we will drop the adjective "convex" and simply refer to $\mathcal{Q}$ as a *polyhedron*.

So, $\mathcal{Q} \subseteq \mathbb{R}^d$ is a polyhedron if there are hyperplanes $H_i = \{\mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{a}_i\,\mathbf{x} = b_i\}$ for $i = 1, 2, \ldots, k$ such that

$$\mathcal{Q} \;=\; \bigcap_{i=1}^{k} H_i^{\leq} \;=\; \left\{ \mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{a}_i\,\mathbf{x} \leq b_i \text{ for all } i = 1, 2, \ldots, k \right\}. \tag{3.2}$$

We call a halfspace $H_j^{\leq}$ **irredundant** if $\bigcap_{i \neq j} H_i^{\leq} \neq \mathcal{Q}$. We might as well only use irredundant halfspaces to describe a given polyhedron. By arranging the normals to the hyperplanes as the rows of a matrix $\mathbf{A} \in \mathbb{R}^{k \times d}$ and letting $\mathbf{b} = (b_1, b_2, \ldots, b_k)$, we compactly write

$$\mathcal{Q} = \left\{ \mathbf{x} \in \mathbb{R}^d \,:\, \mathbf{A}\,\mathbf{x} \leq \mathbf{b} \right\}.$$

Trivially, all affine subspaces, including $\mathbb{R}^d$ and $\varnothing$, are polyhedra.

As in all geometric disciplines, a fundamental notion is that of dimension. For polyhedra, this turns out to be pretty straightforward. We define the **affine hull** $\operatorname{aff}(\mathcal{Q})$ of a polyhedron $\mathcal{Q} \subseteq \mathbb{R}^d$ as the smallest affine subspace of $\mathbb{R}^d$ that contains $\mathcal{Q}$. The reader might want to check (in Exercise 3.15) that

$$\operatorname{aff}(\mathcal{Q}) \;=\; \bigcap \{H_i : \mathcal{Q} \subseteq H_i\}. \tag{3.3}$$

The **dimension** of a polyhedron is the dimension of its affine hull. When $\dim \mathcal{Q} = d$, we call $\mathcal{Q}$ a d**-polyhedron**.

The (topological) **interior** of a polyhedron $\mathcal{Q}$ given in the form (3.2) is

$$\left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}_i\,\mathbf{x} < b_i \text{ for all } i = 1, 2, \ldots, k \right\}. \tag{3.4}$$

However, that notion of interior is not intrinsic to $\mathcal{Q}$ but makes reference to the ambient space. For example, a triangle might or might not have an interior depending whether we embed it in $\mathbb{R}^2$ or $\mathbb{R}^3$. Luckily, every polyhedron comes with a canonical embedding into its affine hull and we can define the **relative interior** as the set of points of $\mathcal{Q}$ that are in the interior of $\mathcal{Q}$ relative to its embedding into $\operatorname{aff}(\mathcal{Q})$. We will denote the relative interior of $\mathcal{Q}$ by $\mathcal{Q}^\circ$. When $\mathcal{Q}$ is full dimensional, $\mathcal{Q}^\circ$ is given by (3.4). In the case that $\mathcal{Q}$ is not full dimensional, we have to be a bit more careful (the details are the content of Exercises 3.18 and 3.19): Assuming $\mathcal{Q}$ is given in the form (3.2), let $I(\mathcal{Q}) := \{i \in [n] : \mathbf{a}_i \mathbf{p} = b_i \text{ for all } \mathbf{p} \in \mathcal{Q}\}$. Then

$$\mathcal{Q}^\circ = \{\mathbf{x} \in \mathcal{Q} : \mathbf{a}_i\,\mathbf{x} < b_i \text{ for all } i \notin I(\mathcal{Q})\}.$$

Complementary to the affine hull of a polyhedron $\mathcal{Q}$, we define the **lineality space** $\operatorname{lineal}(\mathcal{Q}) \subseteq \mathbb{R}^d$ to be the inclusion-maximal linear subspace of

$\mathbb{R}^d$ such that there exists $\mathbf{q} \in \mathbb{R}^d$ for which $\mathbf{q} + \text{lineal}(\mathcal{Q}) \subseteq \mathcal{Q}$. It follows from convexity that $\mathbf{p} + \text{lineal}(\mathcal{Q}) \subseteq \mathcal{Q}$ for all $\mathbf{p} \in \mathcal{Q}$; see Exercise 3.21. If $\text{lineal}(\mathcal{Q}) = \{0\}$ we call $\mathcal{Q}$ **pointed** or **line-free**.

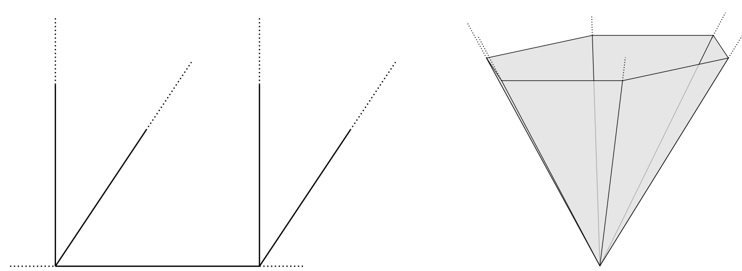

Fig. 3.2 Two polyhedral cones, one of which is pointed.

A set $\mathcal{C} \subseteq \mathbb{R}^d$ is called a **convex cone** if $\mathcal{C}$ is a convex set such that $\mu\mathcal{C} \subseteq \mathcal{C}$ for all $\mu \geq 0$. In particular, every linear subspace is a convex cone and, stronger, any intersection of linear halfspaces is a cone. Intersections of *finitely many* linear halfspaces are called **polyhedral cones**; see Figure 3.2 for two examples and Exercise 3.22 for more. Not all cones are polyhedral: for example,

$$\mathcal{C} \;=\; \left\{(x,y,z) \in \mathbb{R}^3 : z \geq x^2 + y^2\right\}$$

is a cone but not polyhedral (Exercise 3.23). To ease notation, we will typically drop the annotation, as henceforth all the cones to be encountered will be polyhedral. We remark (Exercise 3.24) that a cone $\mathcal{C}$ is pointed if and only if $\mathbf{p}, -\mathbf{p} \in \mathcal{C}$ implies $\mathbf{p} = \mathbf{0}$.

A connection between convex sets and convex cones is the following: For a convex set $\mathcal{K} \subset \mathbb{R}^d$ we define the **homogenization** of $\mathcal{K}$ as

$$\text{hom}(\mathcal{K}) := \left\{(\mathbf{p}, \lambda) \in \mathbb{R}^{d+1} : \lambda \geq 0,\ \mathbf{p} \in \lambda\mathcal{K}\right\}.$$

In particular, for a polyhedron $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ the homogenization is a polyhedral cone given by

$$\text{hom}(\mathcal{Q}) = \left\{(\mathbf{x}, \lambda) \in \mathbb{R}^{d+1} : \lambda \geq 0,\ \mathbf{A}\mathbf{x} \leq \lambda\mathbf{b}\right\}.$$

We can recover our polyhedron $\mathcal{Q}$ from its homogenization as the set of those points $\mathbf{p} \in \text{hom}(\mathcal{Q})$ for which $p_{d+1} = 1$. Moreover, $\text{hom}(\mathcal{Q})$ is pointed if and only if $\mathcal{Q}$ is. Homogenization seems like a simple construction but it will come in handy in this chapter and later ones. The homogenization of a hexagon is shown in Figure 3.2.

As the class of convex sets is closed under taking intersections, there is always a unique inclusion-minimal convex set $\operatorname{conv}(S)$ containing a given set $S \subseteq \mathbb{R}^d$. This set is thus the intersection of all convex sets containing S and is called the **convex hull** of S. For finite S, the resulting objects will play an important role: A **(convex) polytope** $\mathcal{P} \subset \mathbb{R}^d$ is the convex hull of finitely many points in $\mathbb{R}^d$. Thus the polytope defined as the convex hull of $S = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m\}$ is

$$\operatorname{conv}(S) \;=\; \left\{ \lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2 + \cdots + \lambda_m \mathbf{p}_m \;:\; \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_1, \lambda_2, \ldots \lambda_m \geq 0 \right\}.$$

This construction is the essence of "closed under taking line segments"; see Exercise 3.25. Similar to our definition of irredundant halfspaces, we call $\mathbf{v} \in S$ a **vertex** of $\mathcal{P} := \operatorname{conv}(S)$ if $\mathcal{P} \neq \operatorname{conv}(S \setminus \{\mathbf{v}\})$, and we let $\operatorname{vert}(\mathcal{P}) \subseteq S$ denote the collection of vertices of $\mathcal{P}$. This is the unique inclusion-minimal subset $T \subseteq S$ such that $\mathcal{P} = \operatorname{conv}(T)$; see Exercise 3.26.

Likewise, the convex cones are closed under taking intersections, and so we define the **conical hull** $\operatorname{cone}(S)$ of a set $S \subseteq \mathbb{R}^d$ as the smallest convex cone that contains S. The homogenization of a polytope $\mathcal{P}$ is thus given by (Exercise 3.28)

$$\hom(\mathcal{P}) \;=\; \operatorname{cone}(\mathcal{P} \times \{1\}) \;=\; \operatorname{cone}\{(\mathbf{v}, 1) : \mathbf{v} \in \operatorname{vert}(\mathcal{P})\} \tag{3.5}$$

and, as above, the polytope $\mathcal{P}$ can be recovered by intersecting $\hom(\mathcal{P})$ with the hyperplane $H = \{(\mathbf{x}, x_{d+1}) \in \mathbb{R}^{d+1} : x_{d+1} = 1\}$. This relation holds in somewhat greater generality—see Exercise 3.29.

A special class of polytopes consists of the simplices: A d**-simplex** Δ is the convex hull of $d+1$ affinely independent points in $\mathbb{R}^n$. Simplices are the natural generalizations of line segments, triangles, and tetrahedra to higher dimensions.

It turns out that polyhedra and convex/conical hulls of finite sets are two sides of the same coin. To make this more precise, we need the following notion: The **Minkowski sum** of two convex sets $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^d$ is the set

$$\mathcal{K}_1 + \mathcal{K}_2 := \{\mathbf{p} + \mathbf{q} : \mathbf{p} \in \mathcal{K}_1,\ \mathbf{q} \in \mathcal{K}_2\}.$$

An example is depicted in Figure 3.3. That $\mathcal{K}_1 + \mathcal{K}_2$ is again convex is the content of Exercise 3.31. Minkowski sums are key to the following fundamental theorem of polyhedral geometry.

Theorem 3.1 (Minkowski–Weyl). *A set $\mathcal{Q} \subseteq \mathbb{R}^d$ is a polyhedron if and only if there exist a polytope $\mathcal{P}$ and a polyhedral cone $\mathcal{C}$ such that*

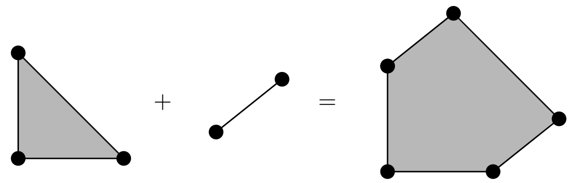

Fig. 3.3 A Minkowski sum.

$$\mathcal{Q} = \mathcal{P} + \mathcal{C}.$$

This theorem highlights the special role of polyhedra among all convex bodies. It states that polyhedra possess a discrete *intrinsic* description in terms of finitely many vertices and generators of $\mathcal{P}$ and $\mathcal{C}$, respectively, as well as a discrete *extrinsic* description in the form of finitely many linear inequalities. The benefit of switching between different presentations is apparent. As a first application we get the following nontrivial operations on polyhedra, polytopes, and cones.

Corollary 3.2. *Let $\mathcal{Q} \subset \mathbb{R}^d$ be a polyhedron and $\phi(\mathbf{x}) = \mathbf{A}\,\mathbf{x} + \mathbf{b}$ an affine projection $\mathbb{R}^d \to \mathbb{R}^e$. Then $\phi(\mathcal{Q})$ is a polyhedron. If $\mathcal{P} \subset \mathbb{R}^d$ is a polytope, then $\mathcal{P} \cap \mathcal{Q}$ is a polytope.*

A hyperplane $H := \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} = b\}$ is a **supporting hyperplane** of the polyhedron $\mathcal{Q}$ if

$$\mathcal{Q} \cap H \neq \varnothing \text{ and either } \mathcal{Q} \subseteq \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \leq b\right\} \text{ or } \mathcal{Q} \subseteq \left\{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} \geq b\right\},$$

in words, $\mathcal{Q}$ is entirely contained in one of the halfspaces bounded by H. A **face** of $\mathcal{Q}$ is a set of the form $\mathcal{Q} \cap H$, where H is a supporting hyperplane of $\mathcal{Q}$; we always include $\mathcal{Q}$ itself (and sometimes $\varnothing$) in the list of faces of $\mathcal{Q}$. The 0-dimensional faces of $\mathcal{Q}$ are precisely the vertices of $\mathcal{Q}$ (Exercise 3.26). The faces of dimension 1 and $d-1$ of a d-dimensional polyhedron $\mathcal{Q}$ are called **edges** and **facets**, respectively. It follows almost by definition that a face of a polyhedron is a polyhedron (Exercise 3.32). An equally sensible but somewhat less trivial fact is that every face of a polyhedron $\mathcal{Q}$ is the intersection of some facets of $\mathcal{Q}$ (Exercise 3.33).

The faces of a given d-polyhedron $\mathcal{Q}$ (including $\varnothing$) are naturally ordered by set containment, which gives rise to a poset (in fact, a lattice), the **face lattice** $\Phi(\mathcal{Q})$. Figure 3.4 gives an example, the face lattice of a square pyramid.

In a sense, much of what remains in this chapter is devoted to face lattices of polyhedra. For starters, we note that $\Phi(\mathcal{Q})$ is naturally graded by dimension,

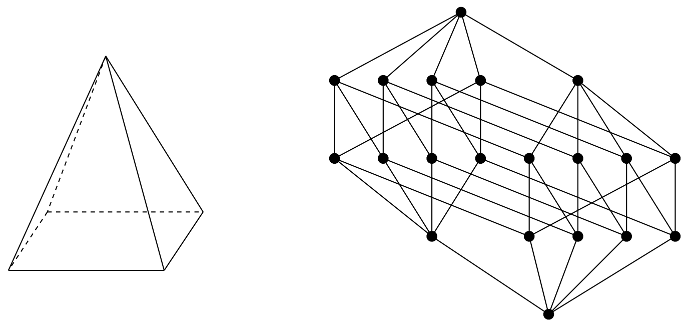

Fig. 3.4 The face lattice of a square pyramid.

which is one of many motivations to introduce the **face numbers**

$$f_k = f_k(\mathcal{Q}) := \#\text{ faces of } \mathcal{Q} \text{ of dimension } k.$$

The face numbers are often recorded in the f**-vector**

$$f(\mathcal{Q}) := (f_0, f_1, \ldots, f_{d-1})$$

It is not hard to see (Exercise 3.27) that the j-faces of a d-polytope $\mathcal{P}$ are in bijection with the $(j+1)$-faces of $\mathrm{hom}(\mathcal{P})$ for all $j = 0, 1, \ldots, d$, and so, in particular,

$$f(\mathrm{hom}(\mathcal{P})) = (1, f(\mathcal{P})).$$

3.2 The Euler Characteristic

We will now come to an important notion that will allow us to relate geometry to combinatorics, the *Euler characteristic*. Our approach to the Euler characteristics of convex polyhedra is by way of sets built up from polyhedra. A set $S \subseteq \mathbb{R}^d$ is **polyconvex** if it is the union of finitely many relatively open polyhedra in $\mathbb{R}^d$:

$$S \;=\; \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_k$$

where $\mathcal{P}_1, \ldots, \mathcal{P}_k \subseteq \mathbb{R}^d$ are relatively open polyhedra. For example, a polyhedron is polyconvex: we can write it as the (disjoint) union of its relatively open faces. Note, however, that our definition entails that, in general, polyconvex sets are not necessarily convex, not necessarily connected, and not necessarily closed. As we will see, they form a good basis of sets to draw from, but not every reasonable set, such as the unit disc in the plane, is a polyconvex set. Let's denote by **PC** the collection of polyconvex sets in $\mathbb{R}^d$. This is an (infinite) poset under inclusion with minimal and maximal elements $\varnothing$ and $\mathbb{R}^d$, respectively. Of course, the intersection and the union of finitely many polyconvex sets is polyconvex, which renders **PC** a distributive lattice.

A map ϕ from **PC** to an Abelian group is a **valuation** if

$$\phi(S \cup T) \;=\; \phi(S) + \phi(T) - \phi(S \cap T) \tag{3.6}$$

for all $S, T \in \mathbf{PC}$. Here is what we're after:

Theorem 3.3. *There exists a unique valuation $\chi : \mathbf{PC} \to \mathbb{Z}$ such that $\chi(\varnothing) = 0$ and $\chi(P) = 1$ for every nonempty polytope $\mathcal{P} \subset \mathbb{R}^d$.*

This is a nontrivial statement as we cannot just define $\chi(S) = 1$ whenever $S \neq \varnothing$. Indeed, if $\mathcal{P} \subset \mathbb{R}^d$ is a d-polytope and $H = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}\,\mathbf{x} = b\}$ is a hyperplane such that $H \cap \mathcal{P} \neq \varnothing$, then

$$\mathcal{P}_1 \;:=\; \{\mathbf{x} \in \mathcal{P} : \mathbf{a}\,\mathbf{x} < b\} \quad \text{and} \quad \mathcal{P}_2 \;:=\; \{\mathbf{x} \in \mathcal{P} : \mathbf{a}\mathbf{x} \geq b\}\,.$$

are polyconvex sets such that $\mathcal{P}_1 \cap \mathcal{P}_2 = \varnothing$ and thus

$$\chi(\mathcal{P}) \;=\; \chi(\mathcal{P}_1) + \chi(\mathcal{P}_2)\,.$$

Therefore, if χ is the valuation of Theorem 3.3, then we need to have $\chi(\mathcal{P}_1) = 0$. The valuation property will be the key to simplifying the computation of $\chi(S)$ for arbitrary polyconvex sets: If $S = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_k$ where each $\mathcal{P}_i \subseteq \mathbb{R}^d$ is a relatively open polyhedron, then iterating (3.6), we obtain the **inclusion–exclusion** formula

$$\chi(S) \;=\; \sum_i \chi(\mathcal{P}_i) - \sum_{i<j} \chi(\mathcal{P}_i \cap \mathcal{P}_j) + \cdots \;=\; \sum_{\varnothing \neq I \subseteq [k]} (-1)^{|I|-1} \chi(\mathcal{P}_I) \tag{3.7}$$

where $\mathcal{P}_I := \cap_{i \in I} \mathcal{P}_i$. In particular, the value of $\chi(S)$ does not depend on the presentation of S as a union of relatively open polyhedra.

Here is a way to construct polyconvex sets. Let $\mathcal{H} = \{H_1, H_2, \ldots, H_n\}$ be an **arrangement** (i.e., a finite set) of n hyperplanes $H_i = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_i\,\mathbf{x} = b_i\}$ in $\mathbb{R}^d$. An example of an arrangement of 6 hyperplanes in the plane is show in Figure 3.5. Continuing with our definitions in (3.1), for a hyperplane H_i, we denote by

$$H_i^{>} \;:=\; \{\mathbf{x} : \mathbf{a}_i\,\mathbf{x} > b_i\}$$

the open positive halfspace bounded by H. We analogously define $H_i^{<}$ and $H_i^{=} := H_i$. For every $\sigma \in \{<,=,>\}^n$, we get a (possibly empty) relatively

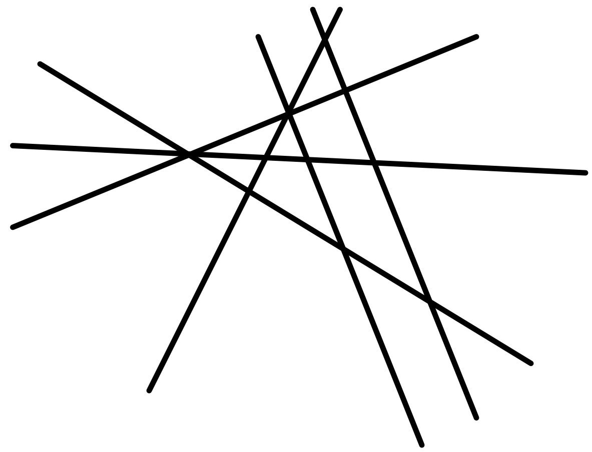

Fig. 3.5 An arrangement of 6 lines in the plane.

open polyhedron

$$H_\sigma \;:=\; H_1^{\sigma_1} \cap H_2^{\sigma_2} \cap \cdots \cap H_n^{\sigma_n}, \tag{3.8}$$

and these relatively open polyhedra partition $\mathbb{R}^d$. For a point $\mathbf{p} \in \mathbb{R}^d$, let $\sigma(\mathbf{p}) \in \{<,=,>\}^d$ record the position of $\mathbf{p}$ relative to the n hyperplanes, that is, $H_{\sigma(\mathbf{p})}$ is the unique relatively open polyhedron among the H_σs containing $\mathbf{p}$. For example, the arrangement in Figure 3.5 decomposes $\mathbb{R}^2$ into 19 regions.

For a fixed hyperplane arrangement $\mathcal{H}$, we define the class of **$\mathcal{H}$-polyconvex sets $\mathbf{PC}(\mathcal{H}) \subset \mathbf{PC}$** as those sets $S \subseteq \mathbb{R}^d$ that are finite unions of relatively open polyhedra of the form H_σ given by (3.8). That is, every $S \in \mathbf{PC}(\mathcal{H})$ has a representation

$$S \;=\; H_{\sigma^1} \uplus H_{\sigma^2} \uplus \cdots \uplus H_{\sigma^k} \tag{3.9}$$

for some $\sigma^1, \sigma^2, \ldots, \sigma^k \in \{<,=,>\}^n$ such that $H_{\sigma^j} \neq \varnothing$ for all $j = 1,2,\ldots,k$. Note that the relatively open polyhedra H_{σ^j} are disjoint and thus the representation of S given in (3.9) is unique. For $S \in \mathbf{PC}(\mathcal{H})$ we define

$$\chi(\mathcal{H},S) \;:=\; \sum_{j=1}^{k} (-1)^{\dim H_{\sigma^j}}. \tag{3.10}$$

The next result (whose proof we leave as Exercise 3.35), states that this function is a valuation.

Proposition 3.4. *The function $\chi(\mathcal{H},\cdot) : \mathbf{PC}(\mathcal{H}) \to \mathbb{Z}$ is a valuation.*

We can consider $\chi(\mathcal{H}, S)$ as a function in two arguments, the arrangement $\mathcal{H}$ and the set $S \subseteq \mathbb{R}^d$. This comes from the property that sets S are typically polyconvex with respect to various arrangements. However, it is a priori not clear how the value of $\chi(\mathcal{H}, S)$ changes when we change the arrangement. The power of our above definition is that it doesn't:

Lemma 3.5. *Let $\mathcal{H}_1, \mathcal{H}_2$ be two hyperplane arrangements in $\mathbb{R}^d$ and let $S \in \mathbf{PC}(\mathcal{H}_1) \cap \mathbf{PC}(\mathcal{H}_2)$. Then*

$$\chi(\mathcal{H}_1, S) \; = \; \chi(\mathcal{H}_2, S).$$

Proof. We note that it is sufficient to show that

$$\chi(\mathcal{H}_1, S) = \chi(\mathcal{H}_1 \cup \{H\}, S) \qquad \text{where} \qquad H \in \mathcal{H}_1 \setminus \mathcal{H}_2. \tag{3.11}$$

Iterating this, we get $\chi(\mathcal{H}_1, S) = \chi(\mathcal{H}_1 \cup \mathcal{H}_2, S)$ and $\chi(\mathcal{H}_2, S) = \chi(\mathcal{H}_1 \cup \mathcal{H}_2, S)$ which proves the claim.

As a next simplifying measure, observe that it is sufficient to show (3.11) for $S = H_\sigma$. Indeed, from the representation in (3.9), we then get

$$\chi(\mathcal{H}_1, S) \; = \; \chi(\mathcal{H}_1, H_{\sigma^1}) + \chi(\mathcal{H}_2, H_{\sigma^2}) + \cdots + \chi(\mathcal{H}_k, H_{\sigma^k}).$$

Now suppose that $S = H_\sigma \in \mathbf{PC}(\mathcal{H}_1)$ and $H \in \mathcal{H}_1 \setminus \mathcal{H}_2$. There are three cases how S can lie relative to H. The easiest cases are $S \cap H = S$ and $S \cap H = \varnothing$. In both cases S is genuinely a polyconvex set for $\mathcal{H}_1 \cup \{H\}$ and

$$\chi(\mathcal{H}_1 \cup \{H\}, S) = (-1)^{\dim S} = \chi(\mathcal{H}_1, S).$$

The only interesting case is $\varnothing \neq S \cap H \neq S$. Since S is relatively open, $S^< := S \cap H^<$ and $S^> := S \cap H^>$ are both nonempty, relatively open polyhedra of dimension $\dim S$, and $S^= := S \cap H^=$ is relatively open of dimension $\dim S - 1$. Therefore, $S = S^< \uplus S^= \uplus S^>$ is a presentation of S as an element of $\mathbf{PC}(\mathcal{H}_1 \cup \{H\})$ and

$$\begin{aligned}
\chi(\mathcal{H}_1 \cup \{H\}, S) \; &= \; \chi(\mathcal{H}_1 \cup \{H\}, S^<) + \chi(\mathcal{H}_1 \cup \{H\}, S^=) + \chi(\mathcal{H}_1 \cup \{H\}, S^>) \\
&= \; (-1)^{\dim S} + (-1)^{\dim S - 1} + (-1)^{\dim S} \\
&= \; \chi(\mathcal{H}_1, S)
\end{aligned}$$

proves the claim. □

The argument used in our proof is typical when working with valuations. The valuation property (3.6) allows us to refine polyconvex sets by cutting them with hyperplanes and halfspaces. Clearly, there is no finite subset of hyperplanes $\mathcal{H}$ such that $\mathbf{PC} = \mathbf{PC}(\mathcal{H})$, but as long as we only worry about finitely many polyconvex sets at a time, we can restrict ourselves to $\mathbf{PC}(\mathcal{H})$ for some $\mathcal{H}$.

Proposition 3.6. *Let $S \in \mathbf{PC}$ be a polyconvex set. Then there is a hyperplane arrangement $\mathcal{H}$ such that $S \in \mathbf{PC}(\mathcal{H})$.*

Proof. This should be intuitively clear. We can write $S = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_k$ where $\mathcal{P}_i$ are a relatively open polyhedra. Now for every $\mathcal{P}_i$ there is a finite set of hyperplanes $\mathcal{H}_i := \{H_1, H_2, \ldots, H_m\}$ such that $\mathcal{P}_i = \bigcap_j H_j^{\sigma_j}$ for some $\sigma \in \{<,=,>\}^m$. Thus, $\mathcal{P}_i \in \mathbf{PC}(\mathcal{H}_i)$ and by refining we get $S \in \mathbf{PC}(\mathcal{H}_1 \cup \mathcal{H}_2 \cup \cdots \cup \mathcal{H}_k)$. $\square$

We can express the content of Proposition 3.6 more conceptual. For two hyperplane arrangements $\mathcal{H}_1$ and $\mathcal{H}_2$,

$$\mathcal{H}_1 \subseteq \mathcal{H}_2 \quad \Longrightarrow \quad \mathbf{PC}(\mathcal{H}_1) \subseteq \mathbf{PC}(\mathcal{H}_2).$$

This gives us the first half of Theorem 3.3:

Proposition 3.7. *There is unique valuation $\chi : \mathbf{PC} \to \mathbb{Z}$ such that for $S \in \mathbf{PC}$,*

$$\chi(S) \ = \ \chi(\mathcal{H}, S)$$

for all hyperplane arrangements $\mathcal{H}$ for which $S \in \mathbf{PC}(\mathcal{H})$.

Proof. For a given $S \in \mathbf{PC}$, we define $\chi(S) := \chi(\mathcal{H}, S)$ for any $\mathcal{H}$ such that $S \in \mathbf{PC}(\mathcal{H})$. By Proposition 3.6 this is well defined. For uniqueness, observe that $\chi(\mathcal{P}) = (-1)^{\dim \mathcal{P}}$ for all relatively open polyhedra $\mathcal{P}$ and by refinement, we can write any $S \in \mathbf{PC}$ as a disjoint union of finitely many relatively open polyhedra. This also shows the inclusion–exclusion property. $\square$

Looking back at (3.10), Proposition 3.7 immediately implies:

Corollary 3.8. *If $\mathcal{P}$ is a relatively open polyhedron then $\chi(\mathcal{P}) = (-1)^{\dim \mathcal{P}}$.*

What is left to show is that $\chi(\mathcal{P}) = 1$ whenever $\mathcal{P}$ is a (nonempty) polytope. Let's first note that (3.10) gives us an effective way to compute the Euler characteristic of a polyhedron: if $\mathcal{Q}$ is a polyhedron, then $\mathcal{Q}$ is the disjoint union of the relative interiors of its faces, that is,

$$\mathcal{Q} \ = \ \biguplus_{\substack{F \subseteq \mathcal{Q} \\ F \text{ face}}} F^\circ$$

and

$$\chi(\mathcal{Q}) \;=\; \sum_{F \neq \varnothing \text{ face}} (-1)^{\dim F} \;=\; \sum_{i=0}^{\dim \mathcal{Q}} (-1)^i f_i(\mathcal{Q}) \,. \tag{3.12}$$

This is called the **Euler–Poincaré formula**. From it we instantly obtain

$$\chi(\varnothing) = 0 \qquad \text{and} \qquad \chi(\mathbb{R}^d) = (-1)^d .$$

Recall that the homogenization of $\mathcal{P}$ is the pointed, polyhedral cone $\operatorname{hom}(\mathcal{P}) \subset \mathbb{R}^{d+1}$ with the property that the i-faces of $\mathcal{P}$ are in bijection with $(i+1)$-faces of $\operatorname{hom}(\mathcal{P})$ for all $i = 0, 1, \dots, d$. Hence

$$\chi(\operatorname{hom}(\mathcal{P})) \;=\; \sum_{F \subseteq \operatorname{hom}(\mathcal{P})} (-1)^{\dim F} \;=\; 1 - \sum_{F \subseteq \mathcal{P}} (-1)^{\dim F} \;=\; 1 - \chi(\mathcal{P}) \,.$$

We therefore complete the proof of Theorem 3.3 by establishing the following statement.

Proposition 3.9. *If $\mathcal{C} \subset \mathbb{R}^{d+1}$ is a full-dimensional polyhedral cone that is not a linear space, then $\chi(\mathcal{C}) = 0$.*

In preparation, we note a special property of polyconvex sets: If $S \in \mathbf{PC}$ then so is $\overline{S} := \mathbb{R}^d \setminus S$. Thus, for a polyconvex set $S \subseteq \mathbb{R}^d$,

$$(-1)^d \;=\; \chi(S) + \chi(\overline{S}) \,,$$

and we may compute the Euler characteristic of $\overline{S}$ if this seems easier. For pointed polyhedral cones this is indeed the case.

Proof of Proposition 3.9. Let

$$\mathcal{C} = \left\{ \mathbf{x} \in \mathbb{R}^{d+1} : \mathbf{a}_i \, \mathbf{x} \leq 0 \text{ for } i = 1, 2, \dots, m \right\}$$

be a polyhedral cone. If $\mathcal{C}$ is full-dimensional and proper, then $m \geq 1$ and there is a point $\mathbf{p} \in \mathcal{C}$ such that $\mathbf{a}_i \, \mathbf{p} < 0$ for all $1 \leq i \leq m$. Let $H_i = \{\mathbf{x} : \mathbf{a}_i \, \mathbf{x} = 0\}$ be the hyperplanes corresponding to the halfspaces that bound $\mathcal{C}$. The complement of $\mathcal{C}$ is the union of the open halfspaces $H_i^>$. Thus inclusion–exclusion gives

$$\chi\left(\mathbb{R}^{d+1} \setminus \mathcal{C}\right) \;=\; \chi\left(H_1^> \cup \cdots \cup H_m^>\right) \;=\; \sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|-1} \chi\left(H_I^>\right)$$

where we set $H_I^> := \bigcap_{i \in I} H_i^>$. As all $H_i^>$ are linear halfspaces, the sets $H_I^>$ are either empty or open polyhedra of dimension $d+1$. But the former can not happen: we have $\mathbf{a}_i(-\mathbf{p}) > 0$ for all $i = 1, 2, \dots, m$ and therefore $-\mathbf{p} \in H_I^>$ for all $I \subseteq [m]$. Thus the right-hand side of the above equation is

$$\sum_{\varnothing \neq I \subseteq [m]} (-1)^{|I|-1}(-1)^{d+1} \;=\; -(-1)^{d+1}\sum_{i=1}^{m}\binom{m}{i}(-1)^{i} \;=\; (-1)^{d+1}.$$

Thus $\chi(\overline{\mathcal{C}}) = \chi(\mathbb{R}^{d+1}) = (-1)^{d+1}$ and hence $\chi(\mathcal{C}) = 0$. □

To make our discussion of Euler characteristics of polyhedra complete, we need to treat two more cases. The easier one is that of polyhedra with a nontrivial lineality space.

Corollary 3.10. *Let $\mathcal{Q}$ be a polyhedron with lineality space $L = \operatorname{lineal}(\mathcal{Q})$, then*

$$\chi(\mathcal{Q}) = (-1)^{\dim L}\chi(\mathcal{Q}/L).$$

This is pretty straightforward considering the relationship between faces and their dimensions of $\mathcal{Q}$ and $\mathcal{Q}/L$; we'll leave the details to Exercise 3.36. A generalization of Proposition 3.9 to general pointed, unbounded polyhedra is as follows.

Corollary 3.11. *Let $\mathcal{Q}$ be a pointed polyhedron. If $\mathcal{Q}$ is unbounded, then $\chi(\mathcal{Q}) = 0$.*

Proof. **Add proof: Euler characteristic is difference of Euler characteristics of two polytopes.** □

Make relation between faces faces of $\mathcal{Q}$ and faces of $\hom(\mathcal{Q})$ clear. In particular, faces of $\hom(\mathcal{Q})$ contained in $\{x_{d+1} = 0\}$ are faces of $\mathcal{Q}$ *at infinity*.

3.3 Möbius Functions of Face Lattices

Euler characteristics are fundamental throughout mathematics. In the context of geometric combinatorics they tie together the combinatorics and the geometry of polyhedra in an elegant way. A first evidence of this is provided by the central result of this section: The Möbius function of the face lattice of a polyhedron can be computed in terms of Euler characteristics.

Theorem 3.12. *Let $\mathcal{Q}$ be a polyhedron with face lattice $\Phi = \Phi(\mathcal{Q})$. For any two faces $F, G \in \Phi$ with $F \subseteq G$,*

$$\mu_\Phi(F,G) \;=\; (-1)^{\dim G - \dim F}.$$

Towards a proof of this result, note that we can safely assume that $\mathcal{Q}$ is pointed: Eliminating the lineality space from $\mathcal{Q}$ leaves the face lattice unchanged while, at the same time,

$$\dim(G/\operatorname{lineal}(\mathcal{Q})) - \dim(F/\operatorname{lineal}(\mathcal{Q})) = \dim G - \dim F.$$

Let $\psi(F,G) := (-1)^{\dim G - \dim F}$, and recall from Section 2.2 that the Möbius function μ_Φ is the inverse of the zeta function ζ_Φ and hence unique. Thus, to prove the claim in Theorem 3.12 that $\mu_\Phi(F,G) = \psi(F,G)$ it is sufficient to show that ψ satisfies the defining relations (2.2) for the Möbius function. That is, we have to show that $\psi(F,F) = 1$ and, for $F \subset G$,

$$\sum_K \psi(K,G) \;=\; 0 \tag{3.13}$$

where the sum is over all faces K of G that contain F. That $\psi(F,F) = 1$ is clear from the definition, so the meat lies in (3.13).

Let's first consider the case where $\mathcal{Q}$ is a pointed polyhedral cone and $F = \{0\}$ is the apex of $\mathcal{Q}$. Any nonempty face G of $\mathcal{Q}$ is itself a pointed cone and (3.13) is a sum over all nonempty faces K of G. We calculate

$$\sum_K (-1)^{\dim G - \dim K} \;=\; (-1)^{\dim G} \sum_K (-1)^{\dim K} \;=\; (-1)^{\dim G} \chi(G),$$

where the last equality stems from the Euler–Poincaré formula (3.12). Since G is pointed and unbounded, Corollary 3.11 asserts that $\chi(G) = 0$, which finishes this special case.

This also proves the case where $\mathcal{Q}$ is a pointed polyhedron and $F = \varnothing$ is the empty face, since we can pass from $\mathcal{Q}$ to $\hom(\mathcal{Q})$: For any nonempty face G of $\mathcal{Q}$, the interval $[\varnothing, G]$ in $\Phi(\mathcal{Q})$ is isomorphic to $[\{0\}, \hom(G)]$ in $\Phi(\hom(\mathcal{Q}))$.

For general F and $\mathcal{Q}$ would like to make the same argument—that (3.13) basically computes the Euler characteristic of a polyhedral cone. To achieve this, we take a route that emphasizes the general geometric idea of approximating geometric objects locally by simpler ones. Namely, we will associate to the face F a polyhedral cone that captures the structure around F.

Let $\mathcal{Q} \subseteq \mathbb{R}^d$ be a polyhedron and $\mathbf{q} \in \mathcal{Q}$. The **tangent cone** of $\mathcal{Q}$ at $\mathbf{q}$ is defined by

$$T_{\mathbf{q}}\mathcal{Q} \;:=\; \{\mathbf{q} + \mathbf{u} \in \mathbb{R}^d : \mathbf{q} + \epsilon\mathbf{u} \in \mathcal{Q} \text{ for some } \epsilon > 0\} \;=\; \mathbf{q} + \operatorname{cone}(\mathcal{Q} - \mathbf{q}).$$

The latter characterization yields that $T_{\mathbf{q}}\mathcal{Q}$ is a translate of $\operatorname{cone}(\mathcal{Q} - \mathbf{q})$ which justifies the name. In particular, $T_{\mathbf{q}}\mathcal{Q}$ is a polyhedron of dimension

$\dim T_{\mathbf{q}}\mathcal{Q} = \dim \mathcal{Q}$. The first thing to note is that the tangent cone only depends on the unique face of $\mathcal{Q}$ that contains $\mathbf{q}$ in its relative interior and is thus geared to capture the local neighborhood of faces.

Proposition 3.13. *Let $\mathcal{Q} \subset \mathbb{R}^d$ be a polyhedron and $F \subset \mathcal{Q}$ a nonempty face. Then*

$$T_{\mathbf{p}}\mathcal{Q} \;=\; T_{\mathbf{q}}\mathcal{Q}$$

for any $\mathbf{p}$ and $\mathbf{q}$ in the relative interior of F.

Proof. Let F be a face of

$$\mathcal{Q} = \left\{ \mathbf{x} \in \mathbb{R}^d : \mathbf{a}_i\mathbf{x} \le b_i \text{ for all } i = 1, 2, \ldots, n \right\},$$

let $\mathbf{p}$ and $\mathbf{q}$ be in the relative interior of F, and let $\mathbf{u} \in \mathbb{R}^d$ such that $\mathbf{p} + \epsilon\mathbf{u} \in \mathcal{Q}$ for some $\epsilon > 0$, that is, $b_i \ge \mathbf{a}_i(\mathbf{p} + \epsilon\mathbf{u})$ for all i. In particular, if $I := \{i : \mathbf{a}_i\mathbf{p} = b_i\}$, then $\mathbf{a}_i\mathbf{u} > 0$ for all $i \in I$.

Now if $\mathbf{q} + \mathbf{u} \notin T_{\mathbf{q}}\mathcal{Q}$, then this means that $\mathbf{q} + \delta\mathbf{u} \notin \mathcal{Q}$ for all $\delta > 0$. This implies that there is some i such that $\mathbf{a}_i\mathbf{q} = b_i$ and $\mathbf{a}_i\mathbf{u} < 0$. Since $\mathbf{p}$ and $\mathbf{q}$ are in the relative interior of F, we have that $\mathbf{a}_i\mathbf{q} = b_i$ for all $i \in I$ which contradicts that $\mathbf{a}_i\mathbf{u} < 0$. □

Proposition 3.13 suggests that we can define the tangent cone of a face F of $\mathcal{Q}$ as $T_F\mathcal{Q} := T_{\mathbf{q}}\mathcal{Q}$ where $\mathbf{q} \in F^\circ$. In particular, the lineality space of $T_F\mathcal{Q}$ is

$$\operatorname{lineal}(T_F\mathcal{Q}) = \operatorname{aff}(F) - \mathbf{q}$$

and $T_F\mathcal{Q}/\operatorname{lineal}(T_F\mathcal{Q})$ is a pointed but unbounded polyhedron except for the case $F = \mathcal{Q}$. Here we have $T_{\mathcal{Q}}\mathcal{Q} = \mathbb{R}^d$.

Our interest in tangent cones comes from the fact that the facial structure of $T_F\mathcal{Q}$ is intimately related to the interval $[F, \mathcal{Q}]$in $\Phi(\mathcal{Q})$:

Lemma 3.14. *Let $\mathcal{Q}$ be a polyhedron and F a k-dimensional face of $\mathcal{Q}$. There is an inclusion-preserving bijection between the l-dimensional faces of $\mathcal{Q}$ that contain F and the $(l-k)$-dimensional faces of $T_F\mathcal{Q}$. In other words, $\Phi(T_F\mathcal{Q}) \setminus \{\varnothing\}$ and $[F, \mathcal{Q}]$ are isomorphic as posets.*

Proof. **Add proof** □

With Lemma 3.14 in hand, we can finally prove Theorem 3.12.

Proof of Theorem 3.12. Let $F \subset G$ be faces of $\mathcal{Q}$. Since the value of $\mu_\Phi(F, G)$ depends only on $[F, G]$, an interval in $\Phi(G)$, we might as well assume that

$G = \mathcal{Q}$. Lemma 3.14 allows us to pass to $T_F\mathcal{Q}$ and deduce

$$\mu_{\Phi(\mathcal{Q})}(F, \mathcal{Q}) \;=\; \mu_{\Phi(T_F\mathcal{Q})}(\text{lineal}(T_F\mathcal{Q}), T_F\mathcal{Q}) \;=\; (-1)^{\dim F}\chi(T_F\mathcal{Q}) \;=\; 0\,,$$

which completes the proof. □

3.4 Notes

The 3-dimensional case of (3.12) was proved by Leonard Euler in 1752 [27, 28]. The full (i.e., higher-dimensional) version of (3.12) was discovered by Ludwig Schläfli in 1852 (though published only in 1902 [60]), but Schläfli's proof implicitly assumed that every polytope is shellable (as did numerous proofs of (3.12) that followed Schläfli's), a fact that was established only in 1971 by Heinz Bruggesser and Peter Mani [19]. The first "airtight" proof of (3.12) (in 1893, using tools from algebraic topology) is due to Henri Poincaré [53] (see also, e.g., [35, Theorem 2.44]).

As already mentioned in Chapter 1, classifying face numbers is a major research problem; in dimension 3 this question is answered by Steinitz's theorem [76] (see also [84, Lecture 4]). The classification question is open in dimension 4. One can also ask a similar question for certain *classes* of polyhedra, e.g., simplicial polytopes, i.e., polytopes all of whose faces are simplices. This gives rise to the famous *Dehn–Sommerville equations* which we will come across in Section **??**.

For (much) more about the wonderful combinatorial world of polyhedra, including proofs of the Minkowski–Weyl Theorem, we recommend [33, 84].

Add more Notes

Exercises

3.15. Prove (3.3), namely, that for a polyhedron given in the form (3.2),

$$\text{aff}(\mathcal{Q}) \;=\; \bigcap \{H_i : \mathcal{Q} \subseteq H_i\}\,.$$

3.16. Let $\mathcal{Q}$ be a polyhedron given in the form (3.2), and consider a face F of $\mathcal{Q}$. Renumber the hyperplanes $H_1, H_2, \dots H_k$ so that $F \subseteq H_j$ for $1 \le j \le m$ and $F \not\subseteq H_j$ for $j > m$, for some index m. Prove that

$$F = \bigcap_{j=1}^{m} H_j \cap \bigcap_{j=m+1}^{k} H_j^{\leq}.$$

3.17. Let $\mathcal{P}$ be the convex hull of the finite set S. Prove that $\mathrm{vert}(\mathcal{P})$ is the unique inclusion-minimal subset $T \subseteq S$ with $\mathcal{P} = \mathrm{conv}(T)$.

3.18. Let $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_i \mathbf{x} \leq b_i \text{ for } i = 1, \ldots, k\}$ be a full-dimensional polyhedron. For $\mathbf{p} \in \mathbb{R}^d$ and $\epsilon > 0$, let $B(\mathbf{p}, \epsilon)$ be the ball of radius ϵ centered at $\mathbf{p}$. A point $\mathbf{p} \in \mathcal{Q}$ is an **interior point** of $\mathcal{Q}$ if $B(\mathbf{p}, \epsilon) \subseteq \mathcal{Q}$ for some $\epsilon > 0$. Show that (3.4) is exactly the set of interior points.

3.19. Let $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_i \mathbf{x} \leq b_i \text{ for } i = 1, \ldots, n\}$ be a polyhedron. For a subset $S \subseteq \mathcal{Q}$ define $I(S) = \{i \in [n] : \mathbf{a}_i \mathbf{p} = b_i \text{ for all } \mathbf{p} \in S\}$.

(a) Show that $\mathrm{aff}(\mathcal{Q}) = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{a}_i \mathbf{x} = b_i \text{ for all } i \in I(\mathcal{Q})\}$.

(b) Let $L = \mathrm{aff}(\mathcal{Q})$. Show that a point $\mathbf{p}$ is in the relative interior of $\mathcal{Q}$ if and only if $B(\mathbf{p}, \epsilon) \cap L \subseteq \mathcal{Q} \cap L$ for some $\epsilon > 0$ (cf. Exercise 3.18).

(c) Show that $\mathcal{Q}^\circ = \{\mathbf{x} \in \mathcal{Q} : \mathbf{a}_i \mathbf{x} < b_i \text{ for all } i \notin I(\mathcal{Q})\}$.

3.20. Let $\mathcal{P} = \mathrm{conv}\{\mathbf{p}_1, \ldots, \mathbf{p}_m\}$ be a polytope. Show that a point $\mathbf{q}$ is in the relative interior of $\mathcal{P}$ if there are $\lambda_1, \ldots, \lambda_m > 0$ such that

$$\mathbf{q} = \lambda_1 \mathbf{p}_1 + \cdots + \lambda_m \mathbf{p}_m \quad \text{and} \quad \lambda_1 + \cdots + \lambda_m = 1.$$

For which polytopes is this condition an "if and only if"?

3.21. Let $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{b}\}$ be a polyhedron. Show that $\mathrm{lineal}(\mathcal{Q}) = \ker(\mathbf{A})$. Infer that $\mathbf{p} + \mathrm{lineal}(\mathcal{Q}) \subseteq \mathcal{Q}$ for all $\mathbf{p} \in \mathcal{Q}$.

The definition of lineality spaces makes sense for arbitrary convex sets $\mathcal{K}$ in $\mathbb{R}^d$. Show that in this more general situation, convexity implies that $\mathbf{p} + \mathrm{lineal}(\mathcal{K}) \subseteq \mathcal{K}$ for all $\mathbf{p} \in \mathcal{K}$.

3.22. Prove that a polyhedron $\mathcal{Q} \subseteq \mathbb{R}^d$ is a cone if and only if $\mathcal{Q} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{A}\,\mathbf{x} \leq \mathbf{0}\}$ for some irredundant matrix $\mathbf{A}$.

3.23. Show that

$$\left\{(x, y, z) \in \mathbb{R}^3 \; : \; z \geq x^2 + y^2\right\}$$

is a cone but not polyhedral.

3.24. Show that a polyhedral cone $\mathcal{C}$ is pointed if and only if $\mathbf{p}, -\mathbf{p} \in \mathcal{C}$ implies $\mathbf{p} = \mathbf{0}$.

3.25. Let $\mathcal{P}$ be the convex hull of $\{\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_m\}$. Prove that

$$\mathcal{P} = \left\{ \lambda_1 \mathbf{p}_1 + \lambda_2 \mathbf{p}_2 + \cdots + \lambda_m \mathbf{p}_m : \sum_{i=1}^{m} \lambda_i = 1,\ \lambda_1, \lambda_2, \dots, \lambda_m \geq 0 \right\}.$$

3.26. Prove that the vertices of a polyhedron $\mathcal{Q}$ are precisely the 0-dimensional faces of $\mathcal{Q}$.

3.27. Show that the i-faces of $\mathcal{P}$ are in bijection with the $(i+1)$-faces of $\operatorname{hom}(\mathcal{P})$ for all $i = 0, 1, \dots, d$.

3.28. Prove (3.5), namely, $\operatorname{hom}(\mathcal{P}) = \operatorname{cone}\{(\mathbf{v}, 1) : \mathbf{v} \in \operatorname{vert}(\mathcal{P})\}$. (*Hint:* you will need Theorem 3.1.)

3.29. Let $\mathcal{C} \subset \mathbb{R}^{d+1}$ be a pointed polyhedral cone and let $R \subset \mathcal{C} \setminus \{\mathbf{0}\}$ be a generating set. Show that, if $H \subset \mathbb{R}^{d+1}$ is a hyperplane such that for every $\mathbf{r} \in R$, there is a $\lambda_{\mathbf{r}} \geq 0$ with $\lambda_{\mathbf{r}} \mathbf{r} \in H$, then $\mathcal{C} \cap H$ is a polytope.

3.30. Let

$$\operatorname{vert}(\mathcal{Q}_1) = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\} \qquad \text{and} \qquad \operatorname{vert}(\mathcal{Q}_2) = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$$

and consider a point $\mathbf{s} + \mathbf{t} \in \mathcal{Q}_1 + \mathcal{Q}_2$; then there are coefficients $\lambda_1, \dots, \lambda_m \geq 0$ and $\mu_1, \mu_2, \dots, \mu_n \geq 0$ such that $\sum_i \lambda_i = \sum_j \mu_j = 1$ and

$$\mathbf{s} + \mathbf{t} = \sum_{i=1}^{m} \lambda_i \mathbf{u}_i + \sum_{j=1}^{n} \mu_j \mathbf{v}_j$$

Now set $\alpha_{ij} = \lambda_i \cdot \mu_j \geq 0$ for $(i, j) \in [m] \times [n]$. Prove that

$$\mathbf{s} + \mathbf{t} = \sum_{(i,j) \in [m] \times [n]} \alpha_{ij} (\mathbf{u}_i + \mathbf{v}_j).$$

3.31. Show that if $\mathcal{K}_1, \mathcal{K}_2 \subset \mathbb{R}^d$ are convex then so is the Minkowski sum $\mathcal{K}_1 + \mathcal{K}_2$.

3.32. Prove that a face of a (bounded) polyhedron is again a (bounded) polyhedron.

3.33. Prove that every face of a polyhedron $\mathcal{P}$ is the intersection of some of the facets of $\mathcal{P}$.

3.34. Prove Corollary /refcor:affprojintpolytopes.

3.35. Prove Proposition 3.4: the function $\chi(\mathcal{H}, \cdot)$ from $\mathbf{PC}(\mathcal{H})$ to $\mathbb{Z}$ is a valuation.

3.36. Prove Corollary 3.10.

Chapter 4

Generating Functions

Everything should be made as simple as possible, but not simpler.
Albert Einstein

We now return to a theme started in Chapter 1: counting functions that are polynomials. Before we can ask about possible interpretations of these counting functions at negative integers—the theme of this book—, we need structural results such as Proposition 1.1, which says that the chromatic polynomial is indeed a polynomial. We hope that we conveyed the message in Chapter 1 that such a structural result can be quite nontrivial for a given counting function. Another example is given by the zeta polynomials of Section 2.3: their definition $Z_\Pi(n) := \zeta_\Pi^n(\hat{0}, \hat{1})$ certainly does not hint at the fact that $Z_\Pi(n)$ is indeed a polynomial. One of our goals in this chapter is to develop machinery that allows us to detect and study polynomials. We will do so alongside introducing several families of counting functions and, naturally, we will discover a number of combinatorial reciprocity theorems along the way.

4.1 Matrix Powers

As a warm-up example, we generalize in some sort the zeta polynomials from Section 2.3. A matrix $\mathbf{A} \in \mathbb{C}^{d \times d}$ is **unipotent** if $\mathbf{A} = \mathbf{I} + \mathbf{B}$ where $\mathbf{I}$ is the $d \times d$ identity matrix and there exists a positive integer k such that $\mathbf{B}^k = \mathbf{0}$ (that is, B is **nilpotent**). The zeta functions from Chapter 2 are our motivating examples of unipotent matrices. Let's recall that, thinking of the zeta function of a poset Π as a matrix, the entry $\zeta_\Pi(\hat{0}, \hat{1})$ was crucial in Chapter 2—the analogous entry in powers of ζ_Π gave rise to the zeta polynomial (emphasis on *polynomial*!)

$$Z_\Pi(n) = \zeta_\Pi^n(\hat{0},\hat{1}).$$

Our first result says that one obtains a polynomial the same way from any unipotent matrix.

Proposition 4.1. *Let $\mathbf{A} \in \mathbb{C}^{d\times d}$ be a unipotent matrix, fix indices $1 \le i,j \le d$, and consider the sequence $f(n) := (\mathbf{A}^n)_{ij}$ formed by the (i,j)-entries of the n^{th} powers of $\mathbf{A}$. Then $f(n)$ agrees with a polynomial in n.*

Proof. We essentially repeat the argument behind (2.1) that gave rise to Proposition 2.4. Suppose $\mathbf{A} = \mathbf{I} + \mathbf{B}$ where $\mathbf{B}^k = \mathbf{0}$. Then

$$f(n) = ((\mathbf{I}+\mathbf{B})^n)_{ij} = \sum_{m=0}^{n} \binom{n}{m} (\mathbf{B}^m)_{ij} = \sum_{m=0}^{k-1} \binom{n}{m} (\mathbf{B}^m)_{ij},$$

which is a polynomial in n. □

We now introduce the main tool of this chapter, the **generating function** of the sequence $f(n)$: the formal power series

$$F(z) := \sum_{n\ge 0} f(n)\, z^n.$$

Let's use the arithmetic in our proof of Proposition 4.1 to compute this generating function:

$$\begin{aligned}
F(z) &= \sum_{n\ge 0} f(n)\, z^n = \sum_{n\ge 0}\sum_{m=0}^{k-1} \binom{n}{m} (\mathbf{B}^m)_{ij}\, z^n = \sum_{m=0}^{k-1} (\mathbf{B}^m)_{ij} \sum_{n\ge 0} \binom{n}{m} z^n \\
&= \sum_{m=0}^{k-1} (\mathbf{B}^m)_{ij} \frac{1}{m!} z^m \sum_{n\ge m} n(n-1)\cdots(n-m+1)\, z^{n-m} \\
&= \sum_{m=0}^{k-1} (\mathbf{B}^m)_{ij} \frac{1}{m!} z^m \left(\frac{d}{dz}\right)^m \frac{1}{1-z} = \sum_{m=0}^{k-1} (\mathbf{B}^m)_{ij} \frac{z^m}{(1-z)^{m+1}} \\
&= \frac{\sum_{m=0}^{k-1} (\mathbf{B}^m)_{ij}\, z^m (1-z)^{k-m-1}}{(1-z)^k},
\end{aligned}$$

a rational function of the form $\frac{h(z)}{(1-z)^k}$ for some polynomial $h(z)$ of degree less than k. In particular, Corollary 4.6 below will give the converse to what we just saw: $f(n)$ is a polynomial of degree k if and only if $F(z)$ is a rational function with denominator $(1-z)^{k+1}$. We'll have more to say about this soon.

4.2 Restricted Partitions

It turns out that the class of counting functions that are polynomials can be slightly widened to include *quasipolynomials*, and as we will see shortly, this extension is natural from the viewpoint of rational generating functions. We start with an example.

An **(integer) partition** of the integer n is a sequence $(a_1 \geq a_2 \geq \cdots \geq a_k \geq 1)$ of nonincreasing positive integers such that

$$n \;=\; a_1 + a_2 + \cdots + a_k \,. \tag{4.1}$$

The numbers $a_1, a_2, \ldots, a_k$ are the **parts** of this partition. For example, $(4,2,1)$ and $(3,2,1,1)$ are partitions of 7.

Partitions have been around since at least Euler's time. They provide a fertile ground for famous theorems (see, e.g., the work of Hardy, Ramanujan, and Rademacher) and open problems (e.g., nobody understands how the *parity* of the number of partitions of n behaves), and they provide a just-as-fertile ground for connections to other areas in mathematics and physics (e.g., Young tableaux, which open the window to representation theory).

Our goal is to enumerate partitions with certain restrictions, which will allow us to prove a combinatorial reciprocity theorem. (The enumeration of unrestricted partitions is the subject of Exercise 4.23, but it will not yield a reciprocity theorem.)

In our first example, we restrict the parts of a partition to a finite set $A := \{a_1, a_2, \ldots, a_d\} \subset \mathbb{Z}_{>0}$, that is, we only allow partitions of the form

$$(a_1, \ldots, a_1, a_2, \ldots, a_2, \ldots, a_d, \ldots, a_d) \,.$$

(It is interesting—and related to topics to appear soon—to allow A to be a *multiset*; see Exercise 4.20.) The **restricted partition function** for A is

$$p_A(n) \;:=\; \#\left\{ (m_1, m_2, \ldots, m_d) \in \mathbb{Z}_{\geq 0}^d : m_1 a_1 + m_2 a_2 + \cdots + m_d a_d = n \right\}.$$

It turns out that the problem of determining $p_A(n)$ becomes easier when we look at all evaluations *at once*, and so we encode the sequence $(p_A(n))_{n \geq 0}$ as the coefficients of the generating function

$$P_A(z) \;:=\; \sum_{n \geq 0} p_A(n)\, z^n$$

One advantage of this (and any other) generating function is that it allows us, in a sense, to manipulate the sequence $(p_A(n))_{n \geq 0}$ by the use of algebra:

$$
\begin{aligned}
P_A(z) &= \sum_{m_1,m_2,\ldots,m_d \ge 0} z^{m_1a_1+m_2a_2+\cdots+m_da_d} \\
&= \left(\sum_{m_1\ge 0} z^{m_1a_1}\right)\left(\sum_{m_2\ge 0} z^{m_2a_2}\right)\cdots\left(\sum_{m_d\ge 0} z^{m_da_d}\right) \qquad (4.2)\\
&= \frac{1}{(1-z^{a_1})(1-z^{a_2})\cdots(1-z^{a_d})},
\end{aligned}
$$

where the last identity comes from the geometric series. To see how the generating function of a counting function helps us understand the latter, let's look at the simplest case when A contains only one integer a. In this case

$$P_{\{a\}}(z) = \frac{1}{1-z^a} = 1 + z^a + z^{2a} + \cdots$$

is the generating function for

$$p_{\{a\}}(n) = \begin{cases} 1 & \text{if } a|n, \\ 0 & \text{otherwise} \end{cases}$$

(as expected from the definition of $p_{\{a\}}(n)$). The counting function $p_{\{a\}}(n)$ is our first example of a **quasipolynomial**, that is, a function $q : \mathbb{Z} \to \mathbb{C}$ of the form

$$q(n) = c_n(n)\,n^d + \cdots + c_1(n)\,n + c_0(n),$$

where $c_0(n), c_1(n), \ldots, c_n(n) : \mathbb{Z} \to \mathbb{C}$ are periodic functions in n. In our example, $p_{\{a\}}(n) = c_0(n)$ where $c_0(n)$ is the periodic function (with period a) that returns 1 if n is a multiple of a and 0 otherwise.

Let's look at another example, namely, when A has two elements. The product structure of the accompanying generating function

$$P_{\{a,b\}}(z) = \frac{1}{(1-z^a)(1-z^b)}$$

means that we can compute

$$p_{\{a,b\}}(n) = \sum_{s=0}^{n} p_{\{a\}}(s)\, p_{\{b\}}(n-s).$$

Note that we are summing a quasipolynomial here, and so $p_{\{a,b\}}(n)$ is again a quasipolynomial by the next proposition, whose proof we leave as Exercise 4.17.

Proposition 4.2. *If $q(n)$ is a quasipolynomial, so is $r(n) := \sum_{s=0}^{n} q(s)$. More generally, if $f(n)$ and $g(n)$ are quasipolynomials, so is their* **convolution**

$$c(n) := \sum_{s=0}^{n} f(s)\, g(n-s)\,.$$

We invite the reader to explicitly compute some examples of restricted partition functions, such as $p_{\{1,2\}}(n)$ (Exercise 4.19). Naturally, we can repeatedly apply Proposition 4.2 to deduce:

Corollary 4.3. *The restricted partition function $p_A(n)$ is a quasipolynomial in n.*

Since $p_A(n)$ is a quasipolynomial, we are free to evaluate it at negative integers. Let's define

$$p_A^{\circ}(n) := \#\left\{ (m_1, m_2, \dots, m_d) \in \mathbb{Z}_{>0}^d : m_1 a_1 + m_2 a_2 + \cdots + m_d a_d = n \right\},$$

the number of restricted partitions of n such that every a_i is used at least once.

Theorem 4.4 (Ehrhart). *If $A = \{a_1, a_2, \dots, a_d\} \subset \mathbb{Z}_{>0}$ then*

$$p_A(-n) \;=\; (-1)^{n-1} p_A(n - a_1 - a_2 - \cdots - a_d) \;=\; (-1)^{n-1} p_A^{\circ}(n)\,.$$

Proof. We first observe that the number of partitions of n in which a_i is used at least once is exactly $p_A(n - a_i)$. Thus, setting $a := a_1 + a_2 + \cdots + a_d$, we see that

$$p_A^{\circ}(n) \;=\; p_A(n-a)\,,$$

and this gives the second equality. To prove the first, we use simple algebra on (4.2) to obtain

$$P_A\left(\frac{1}{z}\right) = \frac{z^{a_1 + a_2 + \cdots + a_d}}{(z^{a_1} - 1)(z^{a_2} - 1)\cdots(z^{a_d} - 1)} = (-1)^d z^a P_A(z)\,,$$

whence

$$\begin{aligned}
\sum_{n \ge 0} p_A(n)\, z^n &= (-1)^d\, z^{-a}\, P_A\left(\frac{1}{z}\right) \\
&= (-1)^d\, z^{-a} \sum_{n \ge 0} p_A(n)\, z^{-n} \\
&\overset{(*)}{=} (-1)^{d-1}\, z^{-a} \sum_{n > 0} p_A(-n)\, z^n \\
&= (-1)^{d-1} \sum_{n \ge 0} p_A(-n-a)\, z^n,
\end{aligned}$$

where $(*)$ follows from Exercise 4.27. (A priori, the last sum should start at $n = 1 - a$, but comparing this sum with the generating function on the left-hand side shows that $p_A(n) = 0$ for $1 - a \le n < 0$.) Equating coefficients proves the claim. □

Clearly, constraining the parts to lie in a given set A is not the only restriction on partitions that one can envision, and we will consider a different kind of restriction in Section 4.4.

4.3 Quasipolynomials

We have just defined a quasipolynomial $q(n)$ as a function $\mathbb{Z} \to \mathbb{C}$ of the form

$$q(n) = c_n(n)\, n^n + \cdots + c_1(n)\, n + c_0(n)\,, \tag{4.3}$$

where $c_0, c_1, \ldots, c_d : \mathbb{Z} \to \mathbb{C}$ are periodic functions in n. The **degree** of $q(n)$ is d (assuming that c_d is not the zero function) and the least common period of $c_0(n), c_1(n), \ldots, c_d(n)$ is the **period** of $q(n)$. Alternatively, for a quasipolynomial $q(n)$, there exist a positive integer k and polynomials $p_0(n), p_1(n), \ldots, p_{k-1}(n)$ such that

$$q(n) \;=\; \begin{cases} q_0(n) & \text{if } n \equiv 0 \bmod k, \\ q_1(n) & \text{if } n \equiv 1 \bmod k, \\ \quad\vdots & \\ q_{k-1}(n) & \text{if } n \equiv k-1 \bmod k. \end{cases}$$

The minimal such k is the period of $q(n)$, and for this minimal k, the polynomials $q_0(n), q_1(n), \ldots, q_{k-1}(n)$ are the **constituents** of $q(n)$. Of course, when $k = 1$, we only need one constituent and the coefficient functions $c_0(n), c_1(n), \ldots, c_d(n)$ are constants, and so $q(n)$ is a **polynomial**. Yet another perspective on quasipolynomials is explored in Exercise 4.25.

As we have seen in (4.2) in the previous section, the quasipolynomials of degree d and period k arising from restricted partitions can be encoded into generating functions that can be expressed as rational functions with a particular denominator. A generating function that can be expressed as a quotient of two polynomials is called a **rational generating function** and the following proposition asserts that the denominators of rational generating functions are the key to detecting quasipolynomials.

Proposition 4.5. *Let $q : \mathbb{Z} \to \mathbb{C}$ be a function with associated generating function*

$$Q(z) \;:=\; \sum_{n\geq 0} q(n)\, z^n .$$

Then $q(n)$ is a quasipolynomial of degree $\leq d$ and period dividing k if and only if

$$Q(z) \;=\; \frac{h(z)}{(1-z^k)^{d+1}}$$

where $h(z)$ is a polynomial of degree at most $k(d+1)-1$.

Apparently (4.2) is not of that form but multiplying numerator and denominator by appropriate terms yields the denominator $(1-z^k)^d$ where $k = \operatorname{lcm}(a_1, a_2, \ldots, a_d)$. For example,

$$P_{\{2,3\}}(t) \;=\; \frac{1}{(1-z^2)(1-z^3)} \;=\; \frac{1+z^2+z^3+z^4+z^5+z^7}{(1-z^6)^2} .$$

For the general recipe to convert (4.2) into a form fitting Proposition 4.5, we refer to Exercise 4.26. In particular, the presentation of $Q(z)$ in Proposition 4.5 is typically not reduced, and by getting rid of common factors it can be seen that $\frac{h(z)}{g(z)}$ gives rise to a quasipolynomial if and only if the zeros of $g(z)$ are roots of unity.

The benefit of having Proposition 4.5 is that it gives a pretty effective way of showing that a function $q : \mathbb{Z} \to \mathbb{C}$ is a quasipolynomial. Let's record the important special case $k = 1$.

Corollary 4.6. *A function $q : \mathbb{Z} \to \mathbb{C}$ is a polynomial of degree $\leq d$ if and only if*

$$Q(z) \;=\; \sum_{n\geq 0} q(n)\, z^n \;=\; \frac{h(z)}{(1-z)^{d+1}}$$

where $h(z)$ is of degree $\leq d$.

The following two gadgets will prove useful in the proof of Proposition 4.5: first, the **Eulerian numbers** $\left\langle {d \atop k} \right\rangle$ defined through

$$\sum_{n\geq 0} n^d z^n \;=\; \frac{1}{(1-z)^{d+1}} \sum_{k=0}^{d} \left\langle {d \atop k} \right\rangle z^k , \qquad (4.4)$$

and second, the **binomial series**

$$\frac{1}{(1-z)^{d+1}} \;=\; \sum_{n\geq 0} \binom{d+n}{d} z^n .$$

Proof of Proposition 4.5. Suppose $q(n) = \sum_{j=0}^{d} c_j(n)\, n^j$ is a quasipolynomial of degree $\leq d$ and period dividing k. We will first consider the case $k = 1$, i.e., $q(n) = \sum_{j=0}^{d} c_j\, n^j$ is a polynomial. Then

$$Q(z) = \sum_{n \geq 0} \sum_{j=0}^{d} c_j\, n^j z^n = \sum_{j=0}^{d} c_j \frac{\sum_{m=0}^{j} \left\langle {j \atop m} \right\rangle z^m}{(1-z)^{j+1}}$$

$$= \frac{\sum_{j=0}^{d} c_j (1-z)^{n-j} \sum_{m=0}^{j} \left\langle {j \atop m} \right\rangle z^m}{(1-z)^{d+1}} = \frac{h(z)}{(1-z)^{d+1}},$$

and we observe that the degree of $h(z)$ is at most d.

For general k, we can find polynomials $q_0(n), q_1(n), \dots, q_{k-1}(n)$ of degree $\leq d$ such that

$$q(n) = \begin{cases} q_0(n) & \text{if } n \equiv 0 \bmod k, \\ q_1(n) & \text{if } n \equiv 1 \bmod k, \\ \vdots & \\ q_{k-1}(n) & \text{if } n \equiv k-1 \bmod k. \end{cases}$$

Thus

$$Q(z) = \sum_{a \geq 0} \sum_{b=0}^{k-1} q(ak+b)\, z^{ak+b} = \sum_{b=0}^{k-1} z^b \sum_{a \geq 0} q_b(ak+b)\, z^{ak},$$

and since $q_b(ak+b)$ is a polynomial in a of degree $\leq d$, we can use our already-proven case to conclude that

$$Q(z) = \sum_{b=0}^{k-1} z^b \frac{h_b(z^k)}{(1-z^k)^{d+1}}$$

for some polynomials $h_b(z)$ of degree $\leq d$. Since $\sum_{b=0}^{k-1} z^b h_b(z^k)$ is a polynomial of degree $\leq k(d+1)-1$, this proves the forward implication of Proposition 4.5.

For the converse implication, suppose $Q(z) = \frac{h(z)}{(1-z^k)^{d+1}}$, where $h(z)$ is a polynomial of degree $\leq k(d+1)-1$, say

$$h(z) = \sum_{m=0}^{k(d+1)-1} c_m\, z^m = \sum_{a=0}^{d} \sum_{b=0}^{k-1} c_{ak+b}\, z^{ak+b}.$$

Then

$$Q(z) = h(z) \sum_{j \geq 0} \binom{d+j}{d} z^{kj} = \sum_{j \geq 0} \sum_{b=0}^{k-1} \sum_{a=0}^{d} c_{ak+b} \binom{d+j}{d} z^{k(j+a)+b}$$
$$= \sum_{j \geq 0} \sum_{b=0}^{k-1} \sum_{a=0}^{d} c_{ak+b} \binom{d+j-a}{d} z^{kj+b} = \sum_{j \geq 0} \sum_{b=0}^{k-1} q_b(kj+b)\, z^{kj+b},$$

where $q_b(kj+b) = \sum_{a=0}^{d} c_{ak+b} \binom{d+j-a}{d}$, a polynomial in j of degree $\leq d$. In other words, $Q(z)$ is the generating function of the quasipolynomial with constituents $q_0(n), q_1(n), \dots, q_{b-1}(n)$. $\square$

Our proof shows that the (linear) transformation from $q(n)$ to $h(z)$ is in essence a change of basis in the vector space of polynomials of degree $\leq d$: writing the constituents of the quasipolynomial $q(n)$ in terms of the standard basis $1, n, n^2, \dots, n^d$, the coefficients of $h(z)$ are precisely the coefficients of these constituents when written in terms of the binomial-coefficient basis $\binom{n}{d}, \binom{n+1}{d}, \dots, \binom{n+d}{d}$.

4.4 Plane Partitions

Our second partition example is the simplest case of a **plane partition**[1]; namely, we will count all ways of writing $n = a_1 + a_2 + a_3 + a_4$ such that the integers $a_1, a_2, a_3, a_4 \geq 0$ satisfy the inequalities

$$\begin{array}{ccc} a_1 & \geq & a_2 \\ \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\ a_3 & \geq & a_4 \,. \end{array} \tag{4.5}$$

(For a general plane partition, this array of inequalities can form a rectangle of any size.) Let $pl(n)$ denote the number of plane partitions of n of the form (4.5). We will compute its generating function

$$Pl(z) \ := \ \sum_{n \geq 0} pl(n)\, z^n \ = \ \sum z^{a_1+a_2+a_3+a_4},$$

where the last sum is over all integers a_1, a_2, a_3, a_4 satisfying (4.5). In the spirit of Chapter 2, let

$$\mathcal{C} := \left\{ (a_1, a_2, a_3, a_4) \in \mathbb{Z}_{\geq 0}^4 : a_1, a_2, a_3, a_4 \text{ satisfy } (4.5) \right\}$$

[1] Note that plane partitions are not partitions in the sense of Section 4.2 but a special case of *P-partitions* which we will study in Section 6.3.

be the collection of all plane partitions.[2] Define

$$\begin{aligned}\mathcal{C}_{23} &:= \{(a_1 \geq a_2 \geq a_3 \geq a_4) \in \mathbb{Z}^4_{\geq 0}\}\\ \mathcal{C}_{32} &:= \{(a_1 \geq a_3 \geq a_2 \geq a_4) \in \mathbb{Z}^4_{\geq 0}\}\end{aligned}$$

and observe that $\mathcal{C}_{23} \cup \mathcal{C}_{32} = \mathcal{C}$ and

$$\mathcal{C}_{23} \cap \mathcal{C}_{32} = \{(a_1 \geq a_2 = a_3 \geq a_4)\} =: \mathcal{C}_{2=3}.$$

We leave it as Exercise 4.28 to verify that the corresponding generating functions are

$$\begin{aligned}Pl_{23}(z) &= Pl_{32}(z) = P_{\{1,2,3,4\}}(z) = \frac{1}{(1-z^4)(1-z^3)(1-z^2)(1-z)} \quad \text{and}\\ Pl_{2=3}(z) &= P_{\{1,3,4\}}(z) = \frac{1}{(1-z^4)(1-z^3)(1-z)},\end{aligned}$$

and therefore

$$Pl(z) = Pl_{23}(z) + Pl_{32}(z) - Pl_{2=3}(z) = \frac{1}{(1-z)(1-z^2)^2(1-z^3)}.$$

Multiplying both the denominator and the numerator by $h(z) = z^{16} + z^{15} + 3z^{14} + 4z^{13} + 7z^{12} + 9z^{11} + 10z^{10} + 13z^9 + 12z^8 + 13z^7 + 10z^6 + 9z^5 + 7z^4 + 4z^3 + 3z^2 + z + 1$ takes the generating function into the range of Proposition 4.5 (can you see why?) and identifies $pl(n)$ as a quasipolynomial. We challenge the reader in Exercise 4.29 to compute this quasipolynomial explicitly.

Just as in our proof of Theorem 4.4, we can observe a simple algebraic relation for $Pl(z)$, namely,

$$Pl\left(\frac{1}{z}\right) = z^8 \, Pl(z).$$

And just as before, this gives rise to a reciprocity relation for the plane-partition counting function:

$$pl(-n) = -pl(n-8).$$

There are many generalizations of $pl(n)$; one is given in Exercise 4.30.

[2] You might think about the (geometric) reason why we name this set $\mathcal{C}$.

4.5 Ehrhart–Macdonald Reciprocity for Simplices

A **lattice simplex** is a simplex whose vertices are in $\mathbb{Z}^d$. All simplices we will consider in this section will be full dimensional. We first extend the lattice-point counting definition of Section 1.4 to higher dimensions: if $\Delta \subset \mathbb{R}^d$, let

$$\operatorname{ehr}_\Delta(n) := \#\left(n\Delta \cap \mathbb{Z}^d\right) = \#\left(\Delta \cap \frac{1}{n}\mathbb{Z}^d\right)$$

denote the number of integer lattice points in the n^{th} dilate of Δ, where n is a positive integer. Our next goal is to prove the following reciprocity theorem, of which Theorem 1.15 was the two-dimensional case.

Theorem 4.7 (Ehrhart). *Suppose Δ is a lattice d-simplex.*

(a) *For positive integers n, the counting function $\operatorname{ehr}_\Delta(n)$ is a polynomial in n whose constant term is* 1.

(b) *When this polynomial is evaluated at negative integers, we obtain*

$$\operatorname{ehr}_\Delta(-n) = (-1)^d \operatorname{ehr}_{\Delta^\circ}(n).$$

We note that there are three statements hidden here:

- There is a polynomial $p(x) \in \mathcal{C}[x]$ such that for positive integers n, $p(n) = \operatorname{ehr}_\Delta(n)$.
- The evaluation of this polynomial at 0 is $p(0) = 1$.
- When p is evaluated at negative integers $-n$, we obtain $p(-n) = (-1)^d \operatorname{ehr}_{\Delta^\circ}(n)$.

The counting function $\operatorname{ehr}_\Delta(n)$ assumes a special form if the lattice simplex

$$\Delta = \operatorname{conv}\{\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_d\}$$

is **unimodular**, that is, the vectors $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \dots, \mathbf{v}_d - \mathbf{v}_0$ form a $\mathbb{Z}$-basis for $\mathbb{Z}^d$ (equivalently, $\det(\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \dots, \mathbf{v}_d - \mathbf{v}_0) = \pm 1$). The following result, which can be proved *without* assuming Theorem 4.7, is subject to Exercise 4.33:

Proposition 4.8. *Let Δ be the convex hull of the origin and the d unit vectors in $\mathbb{R}^d$. Then $\operatorname{ehr}_\Delta(n) = \binom{n+d}{d}$, and this polynomial satisfies Theorem 4.7. More generally, $\operatorname{ehr}_\Delta(n) = \binom{n+d}{d}$ for any unimodular simplex Δ.*

Proof of Theorem 4.7. Suppose $\Delta \subseteq \mathbb{R}^d$ is a lattice d-simplex. We use a technique from Chapter 3: namely, we consider the homogenization of $\mathcal{P}$,

$$\hom(\Delta) = \sum_{\mathbf{v} \text{ vertex of } \Delta} \mathbb{R}_{\geq 0}(\mathbf{v}, 1)$$

by lifting the vertices of Δ into $\mathbb{R}^{d+1}$ onto the hyperplane $x_{d+1} = 1$ and taking the nonnegative span of this lifted version of Δ; see Figure 4.1 for an illustration.

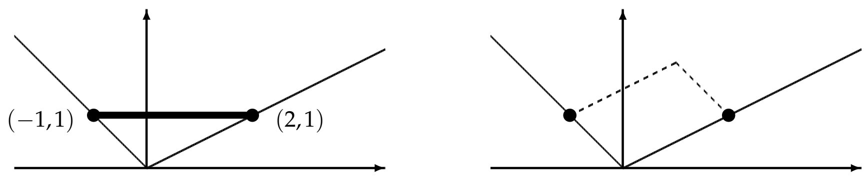

Fig. 4.1 The homogenization of the one-dimensional simplex $[-1,2]$ and its fundamental parallelepiped.

The reason for coning over Δ is that we can see a copy of the dilate $n\Delta$ as the intersection of $\hom(\Delta)$ with the hyperplane $x_{d+1} = n$; we will say that points on this hyperplane are **at height** n. In other words, the **Ehrhart series**

$$\mathrm{Ehr}_\Delta(z) := 1 + \sum_{n>0} \mathrm{ehr}_\Delta(n)\, z^n \tag{4.6}$$

can be computed through

$$\mathrm{Ehr}_\Delta(z) = \sum_{n \geq 0} \#\left(\text{lattice points in } \hom(\Delta) \text{ at height } n\right) z^n. \tag{4.7}$$

We use a tiling argument to compute this generating function. Namely, let

$$\mathcal{Q} := \sum_{\mathbf{v} \text{ vertex of } \Delta} [0,1)(\mathbf{v}, 1),$$

the **fundamental parallelepiped** of $\hom(\Delta)$. Then we can tile $\hom(\Delta)$ by translates of $\mathcal{Q}$, as we invite the reader to prove in Exercise 4.32:

$$\hom(\Delta) = \bigcup_{\mathbf{m} \in \mathbb{Z}_{\geq 0}^{d+1}} \left(\sum_{\mathbf{v} \text{ vertex of } \Delta} m_\mathbf{v}(\mathbf{v}, 1) \; + \; \mathcal{Q} \right), \tag{4.8}$$

where the entries of $\mathbf{m} \in \mathbb{Z}_{\geq 0}^{d+1}$ are indexed by the vertices of Δ, and this union is disjoint.

The vertices of this tiling are all nonnegative integral combinations of the vectors $(\mathbf{v}, 1)$. These vectors are all at height 1, and so their contribution to (4.7) is

$$\sum_{n\geq 0} \#\left(\text{nonnegative integral combinations of } (\mathbf{v},1)\text{'s at height } n\right) z^n = \left(\frac{1}{1-z}\right)^{d+1}.$$

Each other lattice point in $\hom(\Delta)$ is a translate of such a nonnegative integral combination of the $(\mathbf{v},1)$'s by a lattice point in $\mathcal{Q}$. Translated into generating-function language, this gives

$$\begin{aligned}\mathrm{Ehr}_\Delta(z) &= \sum_{n\geq 0} \#\left(\text{lattice points in } \hom(\Delta) \text{ at height } n\right) z^n \\ &= \left(\frac{1}{1-z}\right)^{d+1} \sum_{n\geq 0} \#\left(\text{lattice points in } \mathcal{Q} \text{ at height } n\right) z^n.\end{aligned}$$

So it remains to study

$$h(z) := \sum_{n\geq 0} \#\left(\text{lattice points in } \mathcal{Q} \text{ at height } n\right) z^n.$$

Since $\mathcal{Q}$ has no points at height $\geq d+1$, $h(z)$ is a polynomial of degree at most d. Furthermore, $\mathcal{Q}$ contains the origin, so $h(0) = 1$ and $h(1) = \#\left(\mathcal{Q} \cap \mathbb{Z}^d\right) \geq 1$. This means we can apply Proposition 4.5 to

$$\mathrm{Ehr}_\Delta(z) = \frac{h(z)}{(1-z)^{d+1}}$$

to conclude that $\mathrm{ehr}_\Delta(n)$ is a polynomial. Furthermore,

$$\mathrm{ehr}_\Delta(0) = \mathrm{Ehr}_\Delta(0) = h(0) = 1,$$

which finishes part (a).

Towards part (b), we compute

$$\mathrm{Ehr}_\Delta\left(\frac{1}{z}\right) = \sum_{n\geq 0} \mathrm{ehr}_\Delta(n)\, z^{-n} = \frac{h\left(\frac{1}{z}\right)}{\left(1-\frac{1}{z}\right)^{d+1}} = (-1)^{d+1} \frac{z^{d+1} h\left(\frac{1}{z}\right)}{(1-z)^{d+1}}$$

and so by Exercise 4.27,

$$\sum_{n<0} \mathrm{ehr}_\Delta(n)\, z^{-n} = \sum_{n>0} \mathrm{ehr}_\Delta(-n)\, z^n = (-1)^d \frac{z^{d+1} h\left(\frac{1}{z}\right)}{(1-z)^{d+1}}.$$

Inspired by this, we define

$$\mathrm{Ehr}_{\Delta^\circ}(z) := \sum_{n>0} \mathrm{ehr}_{\Delta^\circ}(n)\, z^n, \tag{4.9}$$

and so proving the reciprocity theorem $\operatorname{ehr}_\Delta(-n) = (-1)^d \operatorname{ehr}_{\Delta^\circ}(n)$ is equivalent to proving

$$\operatorname{Ehr}_{\Delta^\circ}(z) = \frac{z^{d+1} h\left(\frac{1}{z}\right)}{(1-z)^{d+1}}. \tag{4.10}$$

We can compute $\operatorname{Ehr}_{\Delta^\circ}(z)$ along the same lines as we computed $\operatorname{Ehr}_\Delta(z)$ in part (a):

$$\operatorname{Ehr}_{\Delta^\circ}(z) = \sum_{n \geq 0} \#(\text{lattice points in } \hom(\Delta^\circ) \text{ at height } n)\, z^n.$$

The fundamental parallelepiped of $\hom(\Delta^\circ) = \sum_{\mathbf{v} \text{ vertex of } \Delta} \mathbb{R}_{>0}(\mathbf{v},1)$ is

$$\widetilde{\mathcal{Q}} := \sum_{\mathbf{v} \text{ vertex of } \Delta} (0,1](\mathbf{v},1),$$

and

$$\operatorname{Ehr}_{\Delta^\circ}(z) = \frac{\widetilde{h}(z)}{(1-z)^{d+1}},$$

where

$$\widetilde{h}(z) := \sum_{n \geq 0} \#\left(\text{lattice points in } \widetilde{\mathcal{Q}} \text{ at height } n\right) z^n.$$

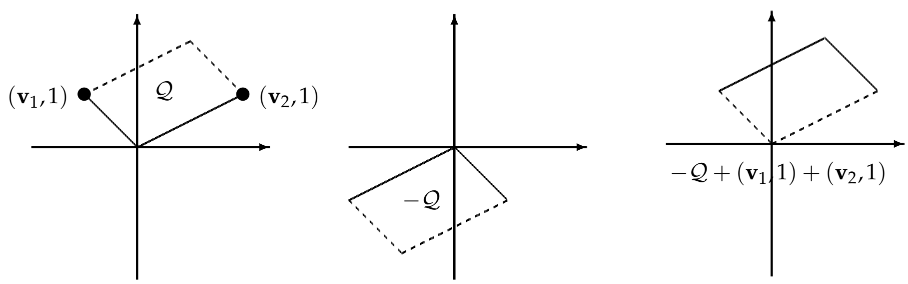

Fig. 4.2 An instance of (4.11).

Fortunately, the parallelepipeds $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ are geometrically closely related, as the reader should work out in Exercise 4.34:

$$\widetilde{\mathcal{Q}} = -\mathcal{Q} + \sum_{\mathbf{v} \text{ vertex of } \Delta} (\mathbf{v},1). \tag{4.11}$$

(Figure 4.2 shows one instance of this relation.) This translates into the generating-function relation

$$\widetilde{h}(z) = h\left(\tfrac{1}{z}\right) z^{d+1}$$

which proves (4.10) and thus part (b). □

This proof can, in fact, be generalized for a **rational simplex**, that is, a simplex that is the convex hull of points in $\mathbb{Q}^d$. We invite the reader to prove the following generalization of Theorem 4.7 in Exercise 4.35:

Proposition 4.9 (Ehrhart). *If Δ is a rational d-simplex, then for positive integers n, the counting function $\operatorname{ehr}_\Delta(n)$ is a quasipolynomial in n whose period divides the least common multiple of the denominators of the vertex coordinates of Δ. When this quasipolynomial is evaluated at negative integers, we obtain*

$$\operatorname{ehr}_\Delta(-n) = (-1)^d \operatorname{ehr}_{\Delta^\circ}(n).$$

Corollary 4.10. *As in (4.6) and (4.9), let*

$$\operatorname{Ehr}_\Delta(z) = \sum_{n \ge 0} \operatorname{ehr}_\Delta(n)\, z^n \qquad \textit{and} \qquad \operatorname{Ehr}_{\Delta^\circ}(z) = \sum_{n > 0} \operatorname{ehr}_{\Delta^\circ}(n)\, z^n$$

denote the generating functions of the Ehrhart quasipolynomials of the d-dimensional rational simplex Δ. Then $\operatorname{Ehr}_\Delta(z)$ and $\operatorname{Ehr}_{\Delta^\circ}(z)$ are rational functions satisfying

$$\operatorname{Ehr}_\Delta\left(\frac{1}{z}\right) = (-1)^{d+1} \operatorname{Ehr}_{\Delta^\circ}(z).$$

Ehrhart–Macdonald Reciprocity holds for general rational polytopes, not just simplices. However, we will need some more machinery before we can prove the general case in Theorem 5.5. Part (a) of Theorem 4.7, on the other hand, *can* be generalized at this point to general rational polytopes, once we introduce triangulations.

Recall that a (convex) polytope $\mathcal{P} = \operatorname{conv}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ is the convex hull of finitely many points $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^d$. If we may choose $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ from $\mathbb{Z}^d$, we call $\mathcal{P}$ a **lattice polytope**. A **triangulation** of a convex d-polytope $\mathcal{P}$ is a finite collection n of d-simplices with the following properties:

- $\mathcal{P} = \bigcup_{\Delta \in T} \Delta$.
- For any $\Delta_1, \Delta_2 \in T$, $\Delta_1 \cap \Delta_2$ is a face of both Δ_1 and Δ_2.

We will see in Chapter 5 that every convex polytope $\mathcal{P}$ can be triangulated with simplices whose vertices are those of $\mathcal{P}$. Assuming this result for the moment, part (a) of Theorem 4.7 yields a famous result:

Theorem 4.11 (Ehrhart). *If $\mathcal{P}$ is a lattice polytope, then for positive integers n, the counting function* $\mathrm{ehr}_{\mathcal{P}}(n)$ *is a polynomial in n whose constant term is* 1.

Proof. Triangulate $\mathcal{P}$ and denote the open faces of the simplices of this triangulation by $\Delta_1^\circ, \Delta_2^\circ, \dots, \Delta_m^\circ$; note that these are all lattice simplices (of varying dimension) with vertices coming from the vertices of $\mathcal{P}$. Since $\mathcal{P}$ is the disjoint union of $\Delta_1^\circ, \Delta_2^\circ, \dots, \Delta_m^\circ$,

$$\mathrm{ehr}_{\mathcal{P}}(n) = \sum_{j=1}^{m} \mathrm{ehr}_{\Delta_j^\circ}(n) .$$

By Theorem 4.7, all the functions on the right-hand side are polynomials, and so $\mathrm{ehr}_{\mathcal{P}}(n)$ is also a polynomial.

To compute its constant term $\mathrm{ehr}_{\mathcal{P}}(0)$, we think of $\mathcal{P} = \Delta_1^\circ \cup \Delta_2^\circ \cup \cdots \cup \Delta_m^\circ$ as a polyconvex set. By Theorem 4.7,

$$\mathrm{ehr}_{\mathcal{P}}(0) = \sum_{j=1}^{m} \mathrm{ehr}_{\Delta_j^\circ}(0) = \sum_{j=1}^{m} (-1)^{\dim \Delta_j} \mathrm{ehr}_{\Delta_j}(0) = \sum_{j=1}^{m} (-1)^{\dim \Delta_j} .$$

But the summands on the right-hand side are the Euler characteristics of $\Delta_1^\circ, \Delta_2^\circ, \dots, \Delta_m^\circ$, by Corollary 3.8. Thus the left-hand side $\mathrm{ehr}_{\mathcal{P}}(0)$ is the Euler characteristic of $\mathcal{P}$, which is 1. □

Naturally, we can repeat the argument in the above proof for **rational polytopes**, that is, polytopes whose vertices are in $\mathbb{Q}^d$, and so Proposition 4.9 yields:

Corollary 4.12 (Ehrhart). *If $\mathcal{P}$ is a rational polytope, then for positive integers n, the counting function* $\mathrm{ehr}_{\mathcal{P}}(n)$ *is a quasipolynomial in n.*

Theorem 4.11 and Corollary 4.12 were proved in 1962 by Eugène Ehrhart, in whose honor we call $\mathrm{ehr}_{\mathcal{P}}(n)$ the **Ehrhart (quasi-)polynomial** of $\mathcal{P}$.

We can also compute Ehrhart quasipolynomials of rational polytopal complexes $\mathcal{C}$: If $\mathrm{ehr}_{\mathcal{C}}(n)$ denotes the number of lattice points in the n-dilate of the union of polytopes in $\mathcal{C}$, then $\mathrm{ehr}_{\mathcal{C}}(n)$ is a quasipolynomial whose constant term is $\mathrm{ehr}_{\mathcal{C}}(0) = \chi(\mathcal{C})$, by repeating the argument given in the last paragraph of our above proof of Theorem 4.11.

4.6 Solid Angles

Suppose $\mathcal{P} \subseteq \mathbb{R}^d$ is a polyhedron. The **solid angle** $\omega_{\mathcal{P}}(\mathbf{x})$ of a point $\mathbf{x}$ (with respect to $\mathcal{P}$) is a real number equal to the proportion of a small ball centered at $\mathbf{x}$ that is contained in $\mathcal{P}$. That is, we let $B_\epsilon(\mathbf{x})$ denote the ball of radius ϵ centered at $\mathbf{x}$ and define

$$\omega_{\mathcal{P}}(\mathbf{x}) := \frac{\operatorname{vol}\left(B_\epsilon(\mathbf{x}) \cap \mathcal{P}\right)}{\operatorname{vol} B_\epsilon(\mathbf{x})}$$

for all sufficiently small ϵ. We note that when $\mathbf{x} \notin \mathcal{P}$, $\omega_{\mathcal{P}}(\mathbf{x}) = 0$; when $\mathbf{x} \in \mathcal{P}^\circ$, $\omega_{\mathcal{P}}(\mathbf{x}) = 1$; and when $\mathbf{x} \in \partial\mathcal{P}$, $0 < \omega_{\mathcal{P}}(\mathbf{x}) < 1$. Computing solid angles is somewhat nontrivial, as the reader can see, e.g., in Exercise 4.40.

Now let $\mathcal{P}$ be a polytope. We define

$$A_{\mathcal{P}}(n) := \sum_{\mathbf{m} \in n\mathcal{P} \cap \mathbb{Z}^d} \omega_{n\mathcal{P}}(\mathbf{m}),$$

the sum of the solid angles at all integer points in $n\mathcal{P}$; recalling that $\omega_{\mathcal{P}}(\mathbf{x}) = 0$ if $\mathbf{x} \notin \mathcal{P}$, we may also write

$$A_{\mathcal{P}}(n) = \sum_{\mathbf{m} \in \mathbb{Z}^d} \omega_{n\mathcal{P}}(\mathbf{m}).$$

The following theorem can be proved along the exact same lines as our proof of Theorem 4.7. We invite the reader to do so in Exercise 4.41.

Theorem 4.13 (Macdonald). *Suppose $\mathcal{P}$ is a lattice d-simplex. Then $A_{\mathcal{P}}(n)$ is a polynomial in n of degree d whose constant term is 0. Furthermore, $A_{\mathcal{P}}(n)$ is either even or odd:*

$$A_{\mathcal{P}}(-n) = (-1)^d A_{\mathcal{P}}(n).$$

Because solid angles behave additively when we glue two polytopes together (and because we do not have to take into account lower-dimensional intersections), this result effortlessly[3] extends to general lattice polytopes, i.e., convex hulls of finitely many points in $\mathbb{Z}^d$:

Corollary 4.14 (Macdonald). *Suppose $\mathcal{P}$ is a lattice d-polytope. Then $A_{\mathcal{P}}(n)$ is a polynomial in n of degree d whose constant term is 0. Furthermore, $A_{\mathcal{P}}(n)$ is either even or odd:*

$$A_{\mathcal{P}}(-n) = (-1)^d A_{\mathcal{P}}(n).$$

[3] As in Section 4.5, here we need the fact that every polytope can be triangulated into simplices, which is the statement of Corollary 5.2.

Note that the analogous step from Theorem 4.13 to Corollary 4.14 is not as easy for Ehrhart polynomials: polynomiality does extend (as we have shown in Section 4.5), but for reciprocity, we need to introduce additional machinery (which we will do in the next chapter). Assuming the general case of Ehrhart–Macdonald Reciprocity (Theorem 5.5) for the moment, we can derive the following classical result, a higher-dimensional analogue of the fact that the angles in a triangle add up to 180°, as the reader should show in Exercise 4.42.

Theorem 4.15 (Brianchon, Gram). *Suppose $\mathcal{P}$ is a rational d-polytope. Then*

$$\sum_{\mathcal{F}} (-1)^{\dim \mathcal{F}} \omega_{\mathcal{P}}(\mathcal{F}) = 0,$$

where the sum is taken over all faces $\mathcal{F}$ of $\mathcal{P}$, and $\omega_{\mathcal{P}}(\mathcal{F}) := \omega_{\mathcal{P}}(\mathbf{x})$ for any $\mathbf{x}$ in the relative interior of $\mathcal{F}$.

4.7 Notes

We have barely started to touch on the useful and wonderful world of generating functions. We heartily recommend [46] and [82] if you'd like to explore more.

Partition functions form a major and long-running theme in number theory; again we barely scratched the surface in this chapter. We recommend [1] and [2] for further study.

Theorem 4.4 is due to Eugène Ehrhart [26]. We will see in the next chapter that it can be vastly generalized to a reciprocity theorem for counting functions that involve not just one but an arbitrary (finite) number of linear constraints.

The formula for the restricted partition function in the case that A contains two elements (given in Exercise 4.21) first appeared, as far as we know, in an 1811 book on elementary number theory by Peter Barlow [7, p. 323–325]. The version we state in Exercise 4.21 seems to go back to a paper by Tiberiu Popoviciu [54], but it has been resurrected at least twice [61, 77].

Restricted partition functions are closely related to a famous problem in combinatorial number theory: namely, what is the largest integer root of $p_A(n)$ (the *Frobenius number* associated with the set A)?[4] This problem,

[4] For this question to make sense, we need to assume that the elements of A are relatively prime.

first raised by Georg Frobenius in the 19th century, is often called the *coin-exchange problem*—it can be phrased in lay terms as looking for the largest amount of money that we cannot change given coin denominations in the set A. Exercise 4.21 suggests that the Frobenius problem is easy for $|A| = 2$ (and you may use this exercise to find a formula for the Frobenius number in this case), but this is deceiving: the Frobenius problem is much harder for $|A| = 3$ (though there exist formulas of sorts [22]) and certainly wide open for $|A| \geq 4$. The Frobenius problem is also interesting from a computational perspective: while the Frobenius number is known to be polynomial-nime computable for fixed $|A|$ [43], implementable algorithms are harder to come by with (see [13] for the current state of the art). For much more on the Frobenius problem, we refer to [55].

The Eulerian numbers defined in (4.4) go back to (surprise!) Leonard Euler and are more commonly defined through the descent set of a permutation. (We will derive this alternative description in Corollary 6.6.) For a bit of history how Euler got interested in these numbers, see [38].

Plane partitions were introduced by Percy MacMahon about a century ago, who proved a famous generating-function formula for the general case of an $m \times n$ plane partition [48]. The formula for 2×2 plane partition diamonds given in Exercise 4.30 is due to George Andrews, Peter Paule, and Axel Riese [3].

Eugène Ehrhart laid the foundation for lattice-point enumeration in rational polyhedra, starting with the proof of Theorem 4.11 (and its rational analogue Corollary 4.12) in 1962 [25] as a teacher at a *lycée* in Strasbourg, France. (Ehrhart received his doctorate later, at age 60 on the urging of some colleagues.) The proof we give here follows Ehrhart's original lines of thought; an alternative proof from first combinatorial principles can be found in [59].

Richard Stanley developed much of the theory of Ehrhart (quasi-)polynomials, initially from a commutative-algebra point of view. One of his famous theorems says that the numerator polynomial of an Ehrhart series has nonnegative integral coefficients [69]. The resulting inequalities serve as the starting point when trying to classify Ehrhart polynomials, though a complete classification is known only in dimension two [9]. The current state of the arts regarding inequalities among Ehrhart coefficients is [74, 75].

I. G. Macdonald inaugurated the systematic study of solid-angle sums in rational polyhedra in 1971 with Corollary 4.14. His paper [47] also contained the first proof of Ehrhart–Macdonald Reciprocity for general rational polytopes (which we give in the next chapter). Some recent results and open questions on solid angles can be found in [11, 23].

The Brianchon–Gram relation (Theorem 4.15) is the solid-angle analogue of the Euler relation (Theorem **??**). It holds for general convex polytopes, even though Theorem 4.15 assumes rational vertices. The 2-dimensional case is ancient (most certainly known to Euclid); the 3-dimensional case was discovered by Charles Julien Brianchon in 1837 [18] and—as far as we know—independently reproved by Jørgen Gram in 1874 [30]. It is not clear who first proved the general d-dimensional case of the Brianchon–Gram relation; the oldest proofs we could find were from the 1960s [33, 51, 63].

For (much) more about Ehrhart (quasi-)polynomials and solid-angle enumeration, see [10], [36], and [72, Chapter 4].

Exercises

4.16. Prove the following extension of Proposition 4.1: Let $\mathbf{A} \in \mathbb{C}^{d\times d}$, fix indices $1 \le i, j \le d$, and consider the sequence $a(n) := (A^n)_{ij}$ formed by the (i,j)-entries of the n^{th} powers of $\mathbf{A}$. Then $a(n)$ agrees with a polynomial in n if and only if $\mathbf{A}$ is unipotent. (*Hint:* Consider the Jordan normal form of $\mathbf{A}$.)

4.17. Prove Proposition 4.2: If $q(n)$ is a quasipolynomial, so is $r(n) := \sum_{s=0}^{n} q(s)$. More generally, if $f(n)$ and $g(n)$ are quasipolynomials then so is their **convolution**

$$c(n) := \sum_{s=0}^{n} f(s)\, g(t-s)\,.$$

4.18. Continuing Exercise 4.17, let $c(n)$ be the convolution of the quasipolynomials $f(n)$ and $g(n)$. What can you say about the degree and the period of $c(n)$, given the degrees and periods of $f(n)$ and $g(n)$?

4.19. Compute the quasipolynomial $p_A(n)$ for the case $A = \{1,2\}$.

4.20. How does your computation of both the generating function and the quasipolynomial $p_A(n)$ change when we switch from Exercise 4.19 to the case of the *multiset* $A = \{1,2,2\}$?

4.21. Suppose a and b are relatively prime positive integers. Define the integers α and β through

$$b\beta \equiv 1 \bmod a \qquad \text{and} \qquad a\alpha \equiv 1 \bmod b\,,$$

and denote by $\{x\}$ the **fractional part** of x, defined through

$$x = \lfloor x \rfloor - \{x\},$$

where $\lfloor x \rfloor$ is the largest integer $\leq x$. Prove that

$$p_{\{a,b\}}(n) = \frac{t}{ab} - \left\{ \frac{\beta t}{a} \right\} - \left\{ \frac{\alpha t}{b} \right\} + 1.$$

4.22. The **(unrestricted) partition function** $p(n)$ counts the number of partitions of n. Show that its generating function is

$$1 + \sum_{n>0} p(n)\, z^n = \prod_{k \geq 1} \frac{1}{1 - z^k}.$$

4.23. Let $d(n)$ denote the number of partitions of n into distinct parts (i.e., no part is used more than once), and let $o(n)$ denote the number of partitions of n into odd parts (i.e., each part is an odd integer). Compute the generating functions of $d(n)$ and $o(n)$, and prove that they are equal (and thus $d(n) = o(n)$ for all positive integers n).

4.24. Compute the constituents and the rational generating function of the quasipolynomial $q(n) = t + (-1)^n$.

4.25. Recall that $\zeta \in \mathbb{C}$ is a root of unity if $\zeta^m = 1$ for some $m \in \mathbb{Z}_{>0}$.

(a) Prove that if $c : \mathbb{Z} \to \mathbb{C}$ is a periodic function with period k, then there are roots of unity $\zeta_0, \zeta_1, \ldots, \zeta_{k-1} \in \mathbb{C}$ and coefficients $c_0, c_1, \ldots, c_{k-1} \in \mathbb{C}$ such that
$$c(n) \;=\; c_0 \zeta_0^n \;+\; c_1 \zeta_1^n \;+\; \cdots \;+\; c_{k-1} \zeta_{k-1}^n.$$

(b) Use this to show that $q : \mathbb{Z}_{\geq 0} \to \mathbb{C}$ is a quasipolynomial if only if
$$q(n) \;=\; \sum_{i=1}^{m} c_i \zeta_i^t n^{k_i}$$
where $c_i \in \mathbb{C}$, $k_i \in \mathbb{Z}_{\geq 0}$, and ζ_i are roots of unity.

4.26. For $A = \{a_1, a_2, \ldots, a_d\} \subset \mathbb{Z}_{>0}$ let $k = \operatorname{lcm}(a_1, a_2, \ldots, a_d)$ be the least common multiple of the elements of A. Provide an explicit polynomial $h_A(z)$ such that generating function $P_A(z)$ for the restricted partitions with respect to A is
$$P_A(z) \;=\; \sum_{n \geq 0} p_A(n) z^n \;=\; \frac{h_A(z)}{(1 - z^k)^d}.$$

4.27. Suppose $q(n)$ is a quasipolynomial. Let

$$Q^+(z) := \sum_{n \geq 0} q(n) \, z^n \qquad \text{and} \qquad Q^-(z) := \sum_{t<0} q(n) \, z^n .$$

Prove that $Q^+(z)$ and $Q^-(z)$ can be written as rational functions that add up to zero: $Q^+(z) + Q^-(z) = 0$.

Here is one way to proceed:

(a) Repeat the proof of Proposition 4.5 to show that $Q^-(z)$ also evaluates to a rational function.

(b) Let $q(n) = 1$. Prove that as rational functions, $Q^+(z) + Q^-(z) = 0$.

(c) Suppose $q(n)$ is a polynomial. Prove that as rational functions, $Q^+(z) + Q^-(z) = 0$.

(d) Suppose $q(n)$ is a quasipolynomial. Prove that as rational functions, $Q^+(z) + Q^-(z) = 0$.

4.28. In this exercise we consider the problem of counting partitions of n with an arbitrary but finite number of parts, restricting the maximal size of each part. That is, define

$$p_{\leq m}(n) \ := \ \#\{(m \geq a_1 \geq a_2 \geq \cdots \geq a_k \geq 1) \, : \, k \in \mathbb{Z}_{>0} \text{ and } a_1 + \cdots + a_k = n\} .$$

Prove that

$$p_{\leq m}(n) \ = \ p_{\{1,2,\ldots,m\}}(n) .$$

4.29. Compute the quasipolynomial $pl(n)$ explicitly.

4.30. Show that the generating function for **plane partition diamonds**

$$\begin{array}{ccccccc}
a_1 & \geq & a_2 & & & & \\
\mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & & & \\
a_3 & \geq & a_4 & \geq & a_5 & & \\
 & & \mathrel{\rotatebox{90}{$\geq$}} & & \mathrel{\rotatebox{90}{$\geq$}} & & \\
 & & a_6 & \geq & a_7 & & \\
 & & & & & \ddots & \\
 & & & & & & a_{3n-2} \geq a_{3n-1} \\
 & & & & & & \mathrel{\rotatebox{90}{$\geq$}} \qquad \mathrel{\rotatebox{90}{$\geq$}} \\
 & & & & & & a_{3n} \quad \geq a_{3n+1}
\end{array}$$

is

$$\frac{(1+z^2)(1+z^5)(1+z^8)\cdots(1+z^{3n-1})}{(1-z)(1-z^2)\cdots(1-z^{3n+1})} .$$

Derive the reciprocity theorem for the associated plane-partition-diamond counting function.

4.31. Pick four concrete points in $\mathbb{Z}^3$ and compute the Ehrhart polynomial of their convex hull.

4.32. Prove (4.8):

$$\hom(\Delta) = \bigcup_{\mathbf{m}\in\mathbb{Z}_{\geq 0}^{d+1}} \left(\sum_{\mathbf{v} \text{ vertex of } \Delta} m_{\mathbf{v}}(\mathbf{v},1) \; + \; \mathcal{Q} \right),$$

and this union is disjoint.

4.33. Prove Proposition 4.8 (without assuming Theorem 4.7): Let Δ be the convex hull of the origin and the d unit vectors in $\mathbb{R}^d$. Then $\operatorname{ehr}_\Delta(n) = \binom{t+d}{d}$, and this polynomial satisfies Theorem 4.7. More generally, $\operatorname{ehr}_\Delta(n) = \binom{t+d}{d}$ for any unimodular simplex Δ.

4.34. Prove (4.11): $\widetilde{\mathcal{Q}} = -\mathcal{Q} + \sum_{\mathbf{v} \text{ vertex of } \Delta} (\mathbf{v},1)\,.$

4.35. Prove Proposition 4.9: If Δ is a rational d-simplex, then for positive integers n, the counting function $\operatorname{ehr}_\Delta(n)$ is a quasipolynomial in n whose period divides the least common multiple of the denominators of the vertex coordinates of Δ. When this quasipolynomial is evaluated at negative integers, we obtain

$$\operatorname{ehr}_\Delta(-n) = (-1)^d \operatorname{ehr}_{\Delta^\circ}(n)\,.$$

4.36. Define the **integer-point transform** of a set $S \subseteq \mathbb{R}^d$ as[5]

$$\sigma_S(z_1,z_2,\dots,z_d) := \sum_{(m_1,m_2,\dots,m_d)\in S\cap\mathbb{Z}^d} z_1^{m_1} z_2^{m_2} \cdots z_d^{m_d}\,.$$

A cone is **simplicial** if its generators are linearly independent.[6] Extend the methods of this section to prove that if $\mathcal{C}$ is a simplicial cone then $\sigma_{\mathcal{C}}(z_1,z_2,\dots,z_d)$ is a rational function which satisfies

$$\sigma_{\mathcal{C}}\left(\tfrac{1}{z_1},\tfrac{1}{z_2},\dots,\tfrac{1}{z_d}\right) = (-1)^{\dim\mathcal{C}} \sigma_{\mathcal{C}^\circ}(z_1,z_2,\dots,z_d)\,.$$

[5] This function also goes by the names of *multivariate generating function* of S and—especially in number theory—*full generating function* of S.

[6] If Δ is a simplex then $\hom(\Delta)$ is an example of a simplicial cone.

4.37. Let Δ be a lattice d-simplex and write

$$\mathrm{Ehr}_{\Delta}(z) = \frac{h_d z^d + h_{d-1} z^{d-1} + \cdots + h_0}{(1-z)^{d+1}}.$$

Prove that

(a) $h_d = \#\left(\Delta^{\circ} \cap \mathbb{Z}^d\right)$;

(b) $h_1 = \#\left(\Delta \cap \mathbb{Z}^d\right) - d - 1$;

(c) $h_0 + h_1 + \cdots + h_d = d!\,\mathrm{vol}(\Delta)$.

4.38. Show that if $\mathcal{P}$ is a d-dimensional lattice polytope in $\mathbb{R}^d$, then the degree of its Ehrhart polynomial $\mathrm{ehr}_{\mathcal{P}}(n)$ is d and the leading coefficient is the volume of $\mathcal{P}$. What can you say if $\mathcal{P}$ is not full dimensional?

4.39. Find and prove an interpretation of the second leading coeffient of $\mathrm{ehr}_{\mathcal{P}}(n)$ for a lattice polytope $\mathcal{P}$. (*Hint:* start by computing the Ehrhart polynomial of the boundary of $\mathcal{P}$.)

4.40. Compute the solid angles of all points in the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. (You may use Theorem 4.15.)

4.41. Prove Theorem 4.13: Suppose $\mathcal{P}$ is a lattice d-simplex. Then $A_{\mathcal{P}}(n)$ is a polynomial in n of degree d whose constant term is 0. Furthermore, $A_{\mathcal{P}}(n)$ is either even or odd:

$$A_{\mathcal{P}}(-n) = (-1)^d A_{\mathcal{P}}(n).$$

4.42. Prove Theorem 4.15: Suppose $\mathcal{P}$ is a rational d-polytope. Then

$$\sum_{\mathcal{F}} (-1)^{\dim \mathcal{F}} \omega_{\mathcal{P}}(\mathcal{F}) = 0,$$

where the sum is taken over all faces $\mathcal{F}$ of $\mathcal{P}$, and $\omega_{\mathcal{P}}(\mathcal{F}) := \omega_{\mathcal{P}}(\mathbf{x})$ for any $\mathbf{x}$ in the relative interior of $\mathcal{F}$.

4.43. State and prove a generating-function analogue of Corollary 4.14 along the lines of Corollary 4.10.

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References

1. George E. Andrews, *The theory of partitions*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1998, Reprint of the 1976 original.
2. George E. Andrews and Kimmo Eriksson, *Integer partitions*, Cambridge University Press, Cambridge, 2004.
3. George E. Andrews, Peter Paule, and Axel Riese, *MacMahon's partition analysis VIII. Plane partition diamonds*, Adv. in Appl. Math. **27** (2001), no. 2-3, 231–242.
4. K. Appel and W. Haken, *Every planar map is four colorable. I. Discharging*, Illinois J. Math. **21** (1977), no. 3, 429–490. MR 0543792 (58 #27598a)
5. K. Appel, W. Haken, and J. Koch, *Every planar map is four colorable. II. Reducibility*, Illinois J. Math. **21** (1977), no. 3, 491–567. MR 0543793 (58 #27598b)
6. Christos A. Athanasiadis, *Characteristic polynomials of subspace arrangements and finite fields*, Adv. Math. **122** (1996), no. 2, 193–233.
7. Peter Barlow, *An Elementary Investigation of the Theory of Numbers*, J. Johnson & Co., London, 1811.
8. Victor V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Algebraic Geom. **3** (1994), no. 3, 493–535, `arXiv:alg-geom/9310003`.
9. Matthias Beck, Jesús A. De Loera, Mike Develin, Julian Pfeifle, and Richard P. Stanley, *Coefficients and roots of Ehrhart polynomials*, Integer Points in Polyhedra—Geometry, Number Theory, Algebra, Optimization, Contemp. Math., vol. 374, Amer. Math. Soc., Providence, RI, 2005, pp. 15–36, `arXiv:math.CO/0402148`.
10. Matthias Beck and Sinai Robins, *Computing the continuous discretely: Integer-point enumeration in polyhedra*, Undergraduate Texts in Mathematics, Springer, New York, 2007, Electronically available at `http://math.sfsu.edu/beck/ccd.html`.
11. Matthias Beck, Sinai Robins, and Steven V Sam, *Positivity theorems for solid-angle polynomials*, Beiträge Algebra Geom. **51** (2010), no. 2, 493–507, `arXiv:0906.4031`.
12. Matthias Beck and Thomas Zaslavsky, *Inside-out polytopes*, Adv. Math. **205** (2006), no. 1, 134–162, `arXiv:math.CO/0309330`.
13. Dale Beihoffer, Jemimah Hendry, Albert Nijenhuis, and Stan Wagon, *Faster algorithms for Frobenius numbers*, Electron. J. Combin. **12** (2005), no. 1, Research Paper 27, 38 pp. (electronic).
14. Garrett Birkhoff, *Lattice theory*, third ed., American Mathematical Society Colloquium Publications, vol. 25, American Mathematical Society, Providence, R.I., 1979.
15. George D. Birkhoff, *A determinant formula for the number of ways of coloring a map*, Ann. of Math. (2) **14** (1912/13), no. 1-4, 42–46.
16. Andreas Blass and Bruce E. Sagan, *Characteristic and Ehrhart polynomials*, J. Algebraic Combin. **7** (1998), no. 2, 115–126.

17. Felix Breuer and Raman Sanyal, *Ehrhart theory, modular flow reciprocity, and the Tutte polynomial*, Math. Z. **270** (2012), no. 1-2, 1–18.
18. Charles J. Brianchon, *Théorème nouveau sur les polyèdres*, J. Ecole (Royale) Polytechnique **15** (1837), 317–319.
19. Heinz Bruggesser and Peter Mani, *Shellable decompositions of cells and spheres*, Math. Scand. **29** (1971), 197–205 (1972).
20. Winfried Bruns and Joseph Gubeladze, *Polytopes, rings, and K-theory*, Springer Monographs in Mathematics, Springer, Dordrecht, 2009.
21. Jesús A. De Loera, Jörg Rambau, and Francisco Santos, *Triangulations*, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010.
22. Graham Denham, *Short generating functions for some semigroup algebras*, Electron. J. Combin. **10** (2003), Research Paper 36, 7 pp. (electronic).
23. David DeSario and Sinai Robins, *A solid angle theory for real polytopes*, to appear in *Quaterly J. Math.*, `arXiv:0708.0042v1`), 2011.
24. Fengming M. Dong, Khee Meng Koh, and Kee L. Teo, *Chromatic polynomials and chromaticity of graphs*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.
25. Eugène Ehrhart, *Sur les polyèdres rationnels homothétiques à n dimensions*, C. R. Acad. Sci. Paris **254** (1962), 616–618.
26. ______, *Sur la partition des nombres*, C. R. Acad. Sci. Paris **259** (1964), 3151–3153.
27. Leonhard Euler, *Demonstatio nonnullarum insignium proprietatum, quibus solida hedris planis inclusa sunt praedita*, Novi Comm. Acad. Sci. Imp. Petropol. **4** (1752/53), 140–160.
28. ______, *Elementa doctrinae solidorum*, Novi Comm. Acad. Sci. Imp. Petropol. **4** (1752/53), 109–140.
29. Dominique Foata, *Distributions eulériennes et mahoniennes sur le groupe des permutations*, Higher Combinatorics (M. Aigner, ed.), NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci., vol. 31, Reidel, Dordrecht, 1977, With a comment by Richard P. Stanley, pp. 27–49.
30. Jørgen P. Gram, *Om rumvinklerne i et polyeder*, Tidsskrift for Math. (Copenhagen) **4** (1874), no. 3, 161–163.
31. Curtis Greene, *Acyclic orientations*, Higher Combinatorics (M. Aigner, ed.), NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci., vol. 31, Reidel, Dordrecht, 1977, pp. 65–68.
32. Curtis Greene and Thomas Zaslavsky, *On the interpretation of Whitney numbers through arrangements of hyperplanes, zonotopes, non-Radon partitions, and orientations of graphs*, Trans. Amer. Math. Soc. **280** (1983), no. 1, 97–126.
33. Branko Grünbaum, *Convex Polytopes*, second ed., Graduate Texts in Mathematics, vol. 221, Springer-Verlag, New York, 2003, Prepared and with a preface by Volker Kaibel, Victor Klee, and Günter M. Ziegler.
34. Philip Hall, *The Eulerian functions of a group*, Quart. J. Math. **7** (1936), 134–151.
35. Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.
36. Takayuki Hibi, *Algebraic Combinatorics on Convex Polytopes*, Carslaw, 1992.
37. ______, *Dual polytopes of rational convex polytopes*, Combinatorica **12** (1992), no. 2, 237–240.
38. Friedrich Hirzebruch, *Eulerian polynomials*, Münster J. Math. **1** (2008), 9–14.
39. June Huh, *Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs*, J. Amer. Math. Soc. **25** (2012), no. 3, 907–927.
40. June Huh and Eric Katz, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*, Math. Ann. **354** (2012), no. 3, 1103–1116. MR 2983081
41. F. Jaeger, *On nowhere-zero flows in multigraphs*, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975) (Winnipeg, Man.), Utilitas Math., 1976, pp. 373–378. Congressus Numerantium, No. XV. MR 0395778 (52 #16570)
42. ______, *Flows and generalized coloring theorems in graphs*, J. Combin. Theory Ser. B **26** (1979), no. 2, 205–216.

43. Ravi Kannan, *Lattice translates of a polytope and the Frobenius problem*, Combinatorica **12** (1992), no. 2, 161–177.
44. Maximilian Kreuzer and Harald Skarke, *Classification of reflexive polyhedra in three dimensions*, Adv. Theor. Math. Phys. **2** (1998), no. 4, 853–871, `arXiv:hep-th/9805190`.
45. ———, *Complete classification of reflexive polyhedra in four dimensions*, Adv. Theor. Math. Phys. **4** (2000), no. 6, 1209–1230, `arXiv:hep-th/0002240`.
46. László Lovász, *Combinatorial Problems and Exercises*, second ed., North-Holland Publishing Co., Amsterdam, 1993.
47. Ian G. Macdonald, *Polynomials associated with finite cell-complexes*, J. London Math. Soc. (2) **4** (1971), 181–192.
48. Percy A. MacMahon, *Combinatory Analysis*, Chelsea Publishing Co., New York, 1960.
49. Peter McMullen, *Lattice invariant valuations on rational polytopes*, Arch. Math. (Basel) **31** (1978/79), no. 5, 509–516.
50. Peter Orlik and Hiroaki Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 300, Springer-Verlag, Berlin, 1992.
51. Micha A. Perles and Geoffrey C. Shephard, *Angle sums of convex polytopes*, Math. Scand. **21** (1967), 199–218.
52. Georg Alexander Pick, *Geometrisches zur Zahlenlehre*, Sitzenber. Lotos (Prague) **19** (1899), 311–319.
53. Henri Poincaré, *Sur la généralisation d'un theorem d'Euler relatif aux polyèdres*, C. R. Acad. Sci. Paris (1893), 144–145.
54. Tiberiu Popoviciu, *Asupra unei probleme de patitie a numerelor*, Acad. Republicii Populare Romane, Filiala Cluj, Studii si cercetari stiintifice **4** (1953), 7–58.
55. Jorge L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications, vol. 30, Oxford University Press, Oxford, 2005.
56. Ronald C. Read, *An introduction to chromatic polynomials*, J. Combinatorial Theory **4** (1968), 52–71.
57. John Riordan, *An introduction to combinatorial analysis*, Dover Publications Inc., Mineola, NY, 2002, Reprint of the 1958 original [Wiley, New York].
58. Gian-Carlo Rota, *On the foundations of combinatorial theory. I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368.
59. Steven V Sam, *A bijective proof for a theorem of Ehrhart*, Amer. Math. Monthly **116** (2009), no. 8, 688–701, `arXiv:0801.4432v5`.
60. Ludwig Schläfli, *Theorie der vielfachen Kontinuität*, Ludwig Schläfli, 1814–1895, Gesammelte Mathematische Abhandlungen, Vol. I, Birkhäuser, Basel, 1950, pp. 167–387.
61. Sinan Sertöz, *On the number of solutions of the Diophantine equation of Frobenius*, Diskret. Mat. **10** (1998), no. 2, 62–71.
62. Paul D. Seymour, *Nowhere-zero 6-flows*, J. Combin. Theory Ser. B **30** (1981), no. 2, 130–135.
63. Geoffrey C. Shephard, *An elementary proof of Gram's theorem for convex polytopes*, Canad. J. Math. **19** (1967), 1214–1217.
64. Neil J. A. Sloane, *On-line encyclopedia of integer sequences*, `http://www.research.att.com/~njas/sequences/index.html`.
65. Eugene Spiegel and Christopher J. O'Donnell, *Incidence algebras*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 206, Marcel Dekker Inc., New York, 1997. MR 1445562 (98g:06001)
66. Richard P. Stanley, *Ordered structures and partitions*, American Mathematical Society, Providence, R.I., 1972, Memoirs of the American Mathematical Society, No. 119.
67. ———, *Acyclic orientations of graphs*, Discrete Math. **5** (1973), 171–178.
68. ———, *Combinatorial reciprocity theorems*, Advances in Math. **14** (1974), 194–253.

69. ______, *Decompositions of rational convex polytopes*, Ann. Discrete Math. **6** (1980), 333–342.
70. ______, *Some aspects of groups acting on finite posets*, J. Combin. Theory Ser. A **32** (1982), no. 2, 132–161. MR 654618 (83d:06002)
71. ______, *Two poset polytopes*, Discrete Comput. Geom. **1** (1986), no. 1, 9–23.
72. ______, *Enumerative Combinatorics. Vol. 1*, Cambridge Studies in Advanced Mathematics, vol. 49, Cambridge University Press, Cambridge, 1997.
73. ______, *An introduction to hyperplane arrangements*, Geometric combinatorics, IAS/Park City Math. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2007, pp. 389–496.
74. Alan Stapledon, *Inequalities and Ehrhart δ-vectors*, Trans. Amer. Math. Soc. **361** (2009), no. 10, 5615–5626, `arXiv:math/0801.0873`.
75. ______, *Additive number theory and inequalities in Ehrhart theory*, Preprint (`arXiv:0904.3035v2`), 2010.
76. Ernst Steinitz, *Polyeder und Raumeinteilungen*, Encyclopädie der mathematischen Wissenschaften, Band 3 (Geometrie), Teil 3AB12 (1922), 1–139.
77. Amitabha Tripathi, *The number of solutions to $ax + by = n$*, Fibonacci Quart. **38** (2000), no. 4, 290–293.
78. W. T. Tutte, *Lectures on matroids*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 1–47.
79. William T. Tutte, *A ring in graph theory*, Proc. Cambridge Philos. Soc. **43** (1947), 26–40.
80. Hassler Whitney, *A logical expansion in mathematics*, Bull. Amer. Math. Soc. **38** (1932), no. 8, 572–579.
81. Herbert S. Wilf, *Which polynomials are chromatic?*, Colloquio Internazionale sulle Teorie Combinatorie (Roma, 1973), Tomo I, Accad. Naz. Lincei, Rome, 1976, pp. 247–256. Atti dei Convegni Lincei, No. 17.
82. ______, *generatingfunctionology*, second ed., Academic Press Inc., Boston, MA, 1994, Electronically available at `http://www.math.upenn.edu/~wilf/`.
83. Thomas Zaslavsky, *Facing up to arrangements: face-count formulas for partitions of space by hyperplanes*, Mem. Amer. Math. Soc. **1** (1975), no. 154.
84. Günter M. Ziegler, *Lectures on polytopes*, Springer-Verlag, New York, 1995, Revised edition, 1998; "Updates, corrections, and more" at `www.math.tu-berlin.de/~ziegler`.

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