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Combinatorial Reciprocity Theorems

Homework # 5 — due February 16

Rules: The maximum number of *regular* points on this homework is **20**. Everything else are bonus points. Thus to safely pass this homework you have to complete (at least) one of the three exercises.

Exercise 1. The cube $C_n = [0,1]^n$ is the order polytope of the n-antichain. The maximal cells of its canonical triangulation Δ are

$$F_\sigma \; = \; \{\phi \in \mathbb{R}^n : 0 \; \leq \; \phi_{\sigma(1)} \; \leq \; \phi_{\sigma(2)} \; \leq \; \cdots \; \leq \; \phi_{\sigma(n-1)} \; \leq \; \phi_{\sigma(n)} \; \leq \; 1\}$$

for all permutations $\sigma \in \mathfrak{S}_n$. In this exercise you will show that this triangulation is *regular*. That is, that there are heights $\omega(v)$ for $v \in \{0,1\}^n$ such that Δ is the complex of bounded faces of

$$\text{conv}\{(v, \omega(v)) \; : \; v \in \{0,1\}^n\} \; + \; \{\lambda \, e_{n+1} : \lambda \geq 0\}.$$

i) Determine the vertices of F_σ.

ii) For a vertex $v \in \{0,1\}^n$ of the cube, define $\omega(v) = |v|(n - |v|)$ where $|v| = \sum_i v_i$. For σ find the unique hyperplane $H_\sigma = \{c^T x + c_{n+1} x_{n+1} \; = \; \delta\}$ such that

$$c^T v \; + \; c_{n+1}\omega(v) \; \geq \; \delta$$

for all $v \in \{0,1\}^n$ and with equality if and only if $v \in F_\sigma$.
[Hint: Consider first the case $\sigma = 123\dots d$ and then use the symmetries of the cube.]

iii) For a general poset P on n-elements argue that $\mathcal{O}(P)$ is the union over all F_σ such that σ is a linear extension of P and, thus, that Δ is a regular subdivision of $\mathcal{O}(P)$.

(3+8+4 points)

Exercise 2. i) Let $Q \subset \mathbb{R}^k$ be a *unimodular* simplex of dimension k. Show that there is no lattice point in the relative interior of $n \cdot Q$ unless $n \geq k+1$.
[Hint: It is sufficient to verify this for your favorite unimodular k-simplex.]
Bonus: What happens if Q is not necessarily unimodular?

(continued on backside)

ii) Verify that a cell $F(\mathcal{I})$ is contained in the interior of $\mathcal{O}(P)$ iff $I_j \setminus I_{j-1}$ is an anti-chain for all $j = 1, \ldots, k+1$.

iii) Show that the following are equivalent

- P has a strict chain of length ℓ.
- $\Omega(P, -k) = 0$ for $k \leq \ell$.
- Δ has no interior cells of dimension $\leq \ell$.

(4+3+8 points)

Exercise 3. Let $G = (V, E)$ be a finite directed graph and define $A \in \{0, -1, 1\}^{V \times E}$ by

$$A_{ve} \;=\; \begin{cases} 1, & \text{if } e = uv, \\ -1, & \text{if } e = vu, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

i) For $I \subseteq E$ denote by G_I the sub-graph spanned by the collection of edges and by A_I the sub-matrix with columns indexed by I. Show that A_I has full column rank $|I|$ if and only if G_I has no cycle.
[Hint: For a cycle, construct and element in the kernel of A_I.]

ii) If G_I is cycle free and b is an integral vector, show that if $A_I\, f = b$ has a solution, then f is integral.
[Hint: If v is a vertex of G_I with only one incident edge $e \in I$, then f_e is $\pm b_v$.]

iii) For $b \in \mathbb{Z}^V$ show that

$$P_G(b) \;=\; \{f \in [0,1]^E : A\, f = b\}$$

is either empty or a lattice polytope. If $p \in P_G(b)$ is a vertex, consider $I = \{i : 0 < p_i < 1\}$.

(5+5+5 points)