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Combinatorial Reciprocity Theorems

Homework # 4 — due ???

Exercise 1. A polyhedron $Q \subseteq \mathbb{R}^d$ is called *pointed* or *line-free* if it does not contain a non-trivial affine subspace (such as a line).

i) Show that if Q is pointed, then there is a hyperplane $H = \{x \in \mathbb{R}^d : c^T x = \delta\}$ such that H meets all unbounded faces and every bounded face is contained in $H^- \setminus H = \{x : c^T x < \delta\}$.
[Hint: To show it first for pointed cones C, meditate on the fact that $\{0\}$ is a face of C.]

ii) Let $Q \subseteq \mathbb{R}^d$ be a d-polyhedron and $U \subseteq \mathbb{R}^d$ be the inclusion-maximal linear subspace such that $p + U \subseteq Q$ for all $p \in Q$. Pick $t \in Q$ and let $U^\perp$ be the orthogonal complement of U. Show that $\widehat{Q} = Q \cap (t + U^\perp)$ is a pointed polyhedron and that the map

$$F \;\mapsto\; \sigma(F) \;=\; F \cap (t + U^\perp)$$

is an order preserving map from $\mathcal{L}(Q)$ to $\mathcal{L}(\widehat{Q})$ that takes k-faces to $k - \dim(Q)$ dimensional faces. [Hint: Try an example first.]

(15 points)

Exercise 2. The *f-vector* of a d-polyhedron Q is $f(Q) = (f_0, f_1, \ldots, f_d)$ with f_i being the number of faces of dimension i.

This exercise tries to get you in touch with software to do computations with polyhedra and their limitations. You can "easily" do the computations using `polymake`[1], `SAGE`[2], ... and hand in a print out of your session.

For $0 \leq k \leq d$ consider the $(d-1)$-polytope $\Delta(d,k) \subset \mathbb{R}^d$ given by the convex hull over all $p \in \{0,1\}^d$ such that p has exactly k ones.

Compute the f-vector of $\Delta(d, \lfloor \frac{d}{2} \rfloor)$ to get $\frac{3^d}{324}$ points for $d = 4, 5, 6, 7$.

Bonus: Can you give a formula for $f_i(\Delta(d,k))$?

[Hint: What is the system of inequalities for $\Delta(d,k)$?]

(10+5 points)

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[1] www.polymake.org, Tutorials:http://polymake.org/doku.php/tutorial/start

[2] www.sagemath.org, Polyhedra in Sage: http://www.sagemath.org/doc/reference/sage/geometry/polyhedra.html

Exercise 3. A poset $\mathcal{P}$ is *graded* if every maximal/saturated chain has the same length. Show that for a polyhedron Q, the face lattice $\mathcal{L}(Q)$ is a graded poset.
[Hint: Induction on the dimension is an option.]

(5 points)