Freie Universität Berlin
Institut für Mathematik
Prof. Raman Sanyal WS2011

Combinatorial Reciprocity Theorems

Homework # 3 — due Dec 5

Exercise 1. i) Let $\sum_{n\geq 0} f(n)t^n = \frac{p(t)}{q(t)}$ be a rational generating function with $q(t) = 1 + \alpha_1 t + \cdots + \alpha_d t^d$ and $\deg p \geq \deg q$. Show that there is a finite *exceptional set* $E \subset \mathbb{N}$ and a function $g : \mathbb{N} \to \mathbb{C}$ such that $g(n) = 0$ for $n \in \mathbb{N} \setminus E$ such that

$$f(n+d) \; + \; \sum_{i=1}^{d} \alpha_i f(n+d-i) \; = \; g(n+d) \; + \; \sum_{i=1}^{d} \alpha_i g(n+d-i)$$

for all $n \in \mathbb{N}$.

ii) Bonus: Consider $\mathbb{N}^n$ as a poset with the componentwise partial order and let $I \subset \mathbb{N}^n$ be a *filter*, i.e. if $a \succeq b \in I$ then $a \in I$. Show that there are $a^1, a^2, \ldots, a^M \in \mathbb{N}^n$ such that

$$I \; = \; \left\{ a \in \mathbb{N}^n \; : \; a \succeq a^i \text{ for some } i = 1, \ldots, M \right\}.$$

[Hint: Consider projections $(b_1, \ldots, b_n) \mapsto (b_1, \ldots, b_{n-1})$.]

iii) Let $I \subseteq \mathbb{N}^n$ be a filter and let

$$f(k) \; = \; \#\{b \in I : b_1 + b_2 + \cdots + b_n = k\}.$$

Show that $f(k)$ is a polynomial for $k \gg 0$ sufficiently large.
[Hint: This is easy if $M = 1$ and $I = \{a \succeq a^1\}$. In case of more generators the existence of joins and inclusion-exclusion might help.]
Bonus: Can you say what the exceptional set is?

(2+4+4+2 points)

Exercise 2. For $k \geq 1$ fixed, let $f_k(n) = n^k$.

i) Show that there are numbers $\alpha_1, \ldots, \alpha_{k+1}$ such that

$$f_k(n+k+1) \; + \; \alpha_1 f_k(n+k) \; + \; \cdots \; + \; \alpha_{k+1} f_k(n) \; = \; 0.$$

ii) This implies that the generating function is rational

$$\sum_{n\geq 0} f_k(n) \; = \; \frac{p_k(t)}{q_k(t)}.$$

What is $q_k(t)$?

iii) **Bonus:** What is $p_k(t)$?
[Hint: Ask Google.]

iv) What is the rational generating function for

$$g_k(n) \;=\; \sum_{i=0}^{n} f_k(i)?$$

(1+1+3+2 points)

Exercise 3. Let $A \in \mathbb{C}^{d\times d}$ be a diagonalizable matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. For fixed $i, j \in [d]$ show that there are $b_1, b_2, \ldots, b_p \in \mathbb{C}$ such that

$$(A^n)_{ij} \;=\; b_1\lambda_1^n \;+\; b_2\lambda_2^n \;+\; \cdots \;+\; b_p\lambda_p^n.$$

[Hint: $A_{ij} = e_i^T A e_j$ where $e_1, \ldots, e_d$ is the standard basis.]

(3 points)

Exercise 4. **Bonus:** Let $f : \mathbb{N} \to \mathbb{C}$ and, for $N \in \mathbb{N}$, let $H_N(f)$ be the *Hankel determinant* $H_N(f) \;=\; \det(f(i+j-2))_{i,j\in[N]}$. E.g. for $N = 4$ this is

$$H_4(f) \;=\; \det\begin{pmatrix} f(0) & f(1) & f(2) & f(3) \\ f(1) & f(2) & f(3) & f(4) \\ f(2) & f(3) & f(4) & f(5) \\ f(3) & f(4) & f(5) & f(6) \end{pmatrix}$$

Show that $\sum_{n\geq 0} f(n)t^n$ is a rational generating function if and only if there is an N_0 such that $H_N(f) = 0$ for all $N \geq N_0$.
[Hint: Show sufficiency ($\Rightarrow$) first.]

(8 points)