FREIE UNIVERSITÄT BERLIN
Institut für Mathematik
Prof. Raman Sanyal

WS2011

Combinatorial Reciprocity Theorems

Homework # 2 — due Thursday, Nov 17

Exercise 1. i) Let P, Q be two posets. Show that every interval of $P \times Q$ is of the form $[a, a']_P \times [b, b']_Q$.

ii) Let P, Q be locally finite. Show that we have

$$\mu_{P\times Q}(I \times J) \;=\; \mu_P(I) \cdot \mu_Q(J)$$

for all intervals $I \times J$ of $P \times Q$.

iii) Show that the Möbius function of the Boolean lattice $\mathcal{B}_k = (2^{[k]}, \subseteq)$ satisfies

$$\mu_{\mathcal{B}_k}(S, T) \;=\; (-1)^{|T\setminus S|}$$

for $S \subseteq T \in \mathcal{B}_k$. [Hint: $\mathcal{B}_k$ is a product.]

iv) For $n \in \mathbb{N}$ fixed consider the map $f_{\leq} : \mathcal{B}_k \to \mathbb{N}$ with

$$f_{\leq}(T) \;=\; |T|^n.$$

Show that

$$k!\, S(n, k) \;=\; (f_{\leq}\, \mu_{\mathcal{B}_k})([k])$$

where $S(n, k)$ is the Stirling number of the second kind.
[Hint: $k!S(n, k)$ counts the number of surjective maps $[n] \to [k]$.]

(1+1+2+4 points)

Exercise 2. Let P be a locally finite poset with zeta function $\zeta = \zeta_P$. For $F \in I(P)$ denote by $F^k = F * F * \cdots * F$ the k-fold product of F in $I(P)$.

i) Show that for $x \preceq y$

$$\zeta^k(x, y) \;=\; \#\{x = x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{k-1} \preceq x_k = y\}$$

counts the number of chains of length k in the interval $[x, y]$.

ii) Show that for $x \preceq y$

$$(\zeta - \delta)^k(x, y) \;=\; \#\{x = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{k-1} \prec x_k = y\}$$

counts the number of *strict* chains of length k in the interval $[x, y]$.

iii) Bonus: For $x, y \in P$, do you know what $(2\delta - \zeta)^{-1}(x, y)$ counts?

(3+4 points)

Exercise 3. The *octahedron* or 3-dimensional *crosspolytope* is the convex polytope

$$\Diamond_3 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : |x_1| + |x_2| + |x_3| \leq 1\}$$

Compute the Ehrhart polynomial of $\Diamond_3$.

Bonus: Can you find a combinatorial interpretation of $E(\Diamond_3, n)$ for $n \in \mathbb{N}$?

(4+4 points)

Exercise 4. i) For distinct points $a, b \in \mathbb{Z}^2$ let $[a, b] = \{(1-\lambda)a + \lambda b \ : \ 0 \leq \lambda \leq 1\}$ be a lattice segment. Show that $E([a, b], t) = g\,t + 1$ where $g = \text{gcd}(|a_1 - b_1|, |a_2 - b_2|)$.

ii) Bonus: Can you show that there is an affine transformation $T(\mathbf{x}) = A\mathbf{x} + c$ with $A \in \mathbb{R}^{2\times 2}$ and $c \in \mathbb{R}^2$ such that $T(\mathbb{Z}^2) = \mathbb{Z}^2$ and $T([a, b]) = [0, \binom{g}{0}]$?

iii) For two numbers $p, q \in \mathbb{Z}_{>0}$ use the above and the Ehrhart polynomial of the special triangle $T = \text{conv}\{\binom{p}{0}, \binom{0}{q}, 0\}$ to show

$$\text{gcd}(p, q) \ = \ 2\sum_{k=1}^{q-1} \left\lfloor \frac{kp}{q} \right\rfloor + p + q - pq$$

(2+4+2 points)