FREIE UNIVERSITÄT BERLIN
Institut für Mathematik
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WS2011

Combinatorial Reciprocity Theorems

Homework # 1 — due Monday, Oct 31

Exercise 1. Find the chromatic polynomials of

i) the wheel with n spokes (and $n+1$ nodes).
For example, the wheel with 6 spokes is .

ii) the cycle on n nodes.

iii) the path on n nodes.

(3+2+1 points)

Exercise 2. Find the chromatic polynomial of the complete bipartite graph $K_{3,3}$.
$K_{n,n}$ has vertices $V = \{1, 2, \ldots, 2n\}$ and edges $E = \{ij \ : \ 1 \leq i \leq n < j \leq 2n\}$.
Bonus: What is it for $K_{n,n}$?

(4+3 points)

Exercise 3. Show that if G has l connected components, then k^l divides the polynomial $\chi_G(k)$.

(2 points)

Exercise 4. Find two simple non-isomorphic graphs G and H with $\chi_G(k) = \chi_H(k)$.
G and H are isomorphic if there is a bijection $\phi : V(G) \to V(H)$ such that

$$uv \in E(G) \quad \Longleftrightarrow \quad \phi(u)\phi(v) \in E(H).$$

Bonus: Can you find many (polynomial, exponential) such examples in the number of vertices?
(Bonus)^2: Can you make your examples arbitrarily high connected?

(3+2+2 points)

(continued on backside)

Exercise 5. Let $G = (V, E)$ be an oriented graph and $k \geq 2$.

i) Let $f : E \to \mathbb{Z}_k$ be a nowhere zero $\mathbb{Z}_k$-flow and let $e \in E$. Show that

$$f : E \setminus \{e\} \to \mathbb{Z}_k$$

is a nowhere zero $\mathbb{Z}_k$-flow on the contraction G/e.

ii) For $S \subseteq V$ let $E^{\text{in}}(S)$ be the *in-coming* edges, i.e., $u \to v$ with $v \in S$ and $u \in V \setminus S$ and $E^{\text{out}}(S)$ the *out-going* edges. Show that $f : E \to \mathbb{Z}_k$ is a nowhere zero $\mathbb{Z}_k$-flow if and only if

$$\sum_{e \in E^{\text{in}}(S)} f(e) \;=\; \sum_{e \in E^{\text{out}}(S)} f(e)$$

for all $S \subseteq V$. [Hint: For the sufficiency contract all edges in S and $V \setminus S$.]

iii) Infer that $\phi_G \equiv 0$ if G has a bridge.

(1+4+1 points)

Exercise 6. Prove that $\phi_G(k)$, the number of nowhere zero $\mathbb{Z}_k$-flows, is a polynomial in k. What is its degree? Show that $\phi_G(k) \leq \phi_G(k+1)$.

(3+2+1 points)