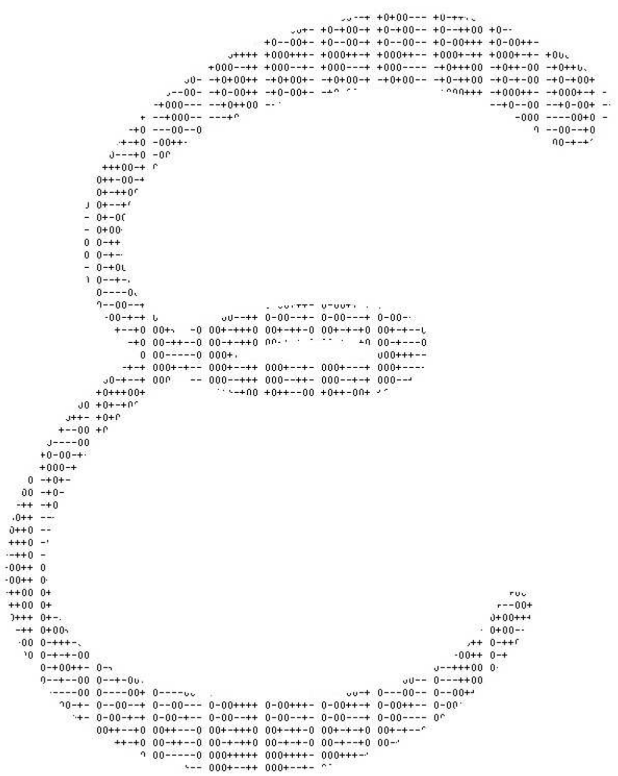
# On the Combinatorics of Projected Deformed Products



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# On the Combinatorics of Projected Deformed Products

Diplomarbeit bei
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an der Fakultät IV (Informatik)
der Technischen Universität Berlin

Berlin, 18. August 2005

Die selbstständige und eigenhändige Anfertigung versichere ich an Eides statt.

Berlin, 18. August 2005

*To my father who introduced me to computers and maths*
*and*
*to my mother who failed to prevent him.*

# Contents

# Introduction

> 'Begin at the beginning, and go on till you come to the end: then stop.'
> —*Lewis Caroll*, from "Alice's Adventures in Wonderland."

Within the realms of combinatorial geometry, polytopes are one of the most fascinating objects to study. One of the reasons for this might be, that polytopes give the impression that one is dealing with "hands-on" geometry. To start with, polytopes in three dimensions are best described as geometric objects with finitely many vertices or, equivalently, as objects bounded by finitely many polygons. Three dimensional polytopes have been around for quite long and still they make people enthusiastic about geometry. Nevertheless, the fields of *interesting* 3-dimensional polytopes are fairly hunted down. As a matter of fact, the classification of 3-polytopes was completed almost a hundred years ago with the work of Ernst Steinitz. So a natural thing to do is to move on in dimension. But in passing to dimension four, imagination inevitably fails. For example, in four dimensional space there exist polytopes with an arbitrary number of vertices with the property that every two vertices are joined by an edge. In 3-space such polytopes are in short supply (the tetrahedron is the only one). This and many examples more suggest that transferring intuitive ideas from 3- to 4-space are insufficient to fathom the unadulterated richness of geometry beyond our imagination.

But, as in any area of mathematical science, the heart beat can be measured by the richness and diversity of ideas. In discrete geometry, however, ingenious constructions

> "if rare in comparison with blackberries, are commoner than returns of Halley's comet."

as G.H. Hardy quotes in his "A Mathematician's Apology". This work is chiefly based on such an ingenious construction as was presented in Ziegler (2004).

The basic idea of the construction is the following. Instead of producing polytopes in $\mathbb{R}^4$ a little detour is taken. Ziegler constructs products of polygons which are high-dimensional polytopes but are easy to analyze. In particular, these products of polygons admit certain deformations that do not alter the combinatorial structure. The key insight now is that even though these deformations retain the combinatorics, projections of this polytopes to 4-space can *look* completely different. In that spirit, Ziegler designs inequality systems corresponding to the afore mentioned polytopes whose projections give rise to 4-polytopes having extremal combinatorial properties. In this work we make an attempt to give a systematic approach to the technique of forming "deformed products" and we are able to give a combinatorial description of the deformed products of polygons.

The work is structured in the following way. In the first chapter we introduce the main actors, namely convex polytopes. Since we will be dealing with nothing else but *convex* polytopes, we will drop the supplement 'convex' henceforth. Polytopes come essentially in two different guises: They are given either as the convex hull of a finite set of points or as an intersection of finitely many halfspaces. A vital part of polytope theory is the liberty to alternate between these two (different) ways of looking at polytopes. We will heavily exercises the right to switch views when we introduce the notion faces of polytopes, resulting in a multitude of ways to describe faces. Faces of a polytope as well as the incidences among them constitute combinatorial data that is commonly associated to a polytope. We introduce the face lattice, a partially ordered set that captures this combinatorial data, as well as a numerical invariant of it, the $f$-vector.

Next, we will introduce the important classes of simple and simplicial polytopes. These polytopes possess the quality that their combinatorial structure is stable under certain perturbations, a quality which lies at the heart of the construction of deformed products of polytopes. This stability is due to certain spatial relations of their facet normals or vertices, respectively, and comes as the ubiquitous concept of "general position". This work, as it is, is just an $\varepsilon$ away from being purely combinatorial. However, this $\varepsilon$ gap manifests itself in the fact that the feasibility of certain deformations relies on metrical properties of the polytopes in question and thus on general positioning. We therefore dedicate some part of the first chapter to a treatment of points being in general position.

Another important class we introduce is that of neighborly polytopes. These are polytopes that exhibit, in a precise sense, extremal behaviour concerning incidences of faces. One of its most valued members is the class of cyclic polytopes, which possess a particularly nice combinatorial description.

Apart from those families of polytopes, there exist several, all in all well understood, techniques of producing new polytopes from old ones. One such technique, to which we devote some time, is that of taking products. As it turns out, taking products of polytopes is a purely combinatorial construction and we will thus elaborate on it in these terms. A well-known family of polytopes along which we will illustrate the product construction is the family of cubes.

Chapter 2 develops all the tools necessary for the construction and analysis of deformed products of polytopes. We start with the introduction of Gale transforms, a tool indispensable in the study of polytopes with few vertices. Gale transforms, which can be stated in terms of basic linear algebra, are a mean of associating to a point configuration a vector configuration that has, in some sense, identical combinatorial properties. In the right setting, this vector configuration lives in a low dimensional space and might even be visualized. This makes it possible to make statements about polytopes that exist beyond human perception. In this work, however, we present a new application of Gale transforms as perturbations of certain polytopes.

A seemingly unrelated topic that we take up in the second chapter is that of subdivisions of polytopes. The basic idea behind subdivisions is that polytopes, or more general geometric objects, can be decomposed into *simpler* building blocks and the object in question can be viewed as the sum of its parts and thus be studied in that spirit. Subdivisions, or more specifically triangulations, are of considerable practical interest. In computer graphics, for instance, surfaces are modelled by sets of triangles that lie edge-to-edge and give, if the triangulation is fine enough, the impression of a smooth object. In the task of modelling solid bodies, the basic building blocks are (combinatorial) cubes and, in computer graphics, these subdivisions go by the name of *hexahedral meshes*. We put emphasis on regular subdivisions and lexicographic subdivisions that arise as projections of polytopal liftings of point configurations. We end the chapter with a way of relating regular subdivisions to Gale transforms by, what we call, perturbed Gale transforms.

Finally, Chapter 3 combines the developed tools in the construction of deformed products. In this last chapter we head for the construction of polytopes with extremal combinatorics. These arise as projections of high dimensional polytopes and therefore we digress on projections of polytopes. We introduce the notion of faces being strictly preserved under projection and give some characterizations of faces that do so.

We proceed by reviewing the notion of "deformed products" as given in Amenta and Ziegler (1999) and introduce possible generalizations. In that light, we review the neighborly cubical polytopes of Joswig and Ziegler (2000) as (generalized) deformed products and retrace their combinatorial description. The construction we propose is more general and leads to many non-isomorphic cubical polytopes in dimensions $d \geq 6$.

Building on neighborly cubical polytopes we reconstruct Ziegler's deformed products of polygons and give, for the first time, a complete combinatorial description of the projection.

**Acknowledgements.** I am grateful to Professor Ziegler for letting me work on the problem and, even more, providing me with such a marvelous working environment within his group. These last month have been a real pleasure for me. I would also like to thank Andreas Paffenholz, Thilo Schröder, Jakob Uszkoreit, Arnold Wassmer, and Axel Werner for their proof-reading and for their endurance to listen to my (mathematical) waffling. In particular, I am very much in debt to Thilo Schröder for many helpful and enlightening discussions.

Last but not least, I like to express my gratitude and deep feelings to Vanessa Kääb for not only being encouraging and supportive in respect to this work, but also for enriching my life with her presence.

# Chapter 1

# Polytope Theory

Throughout this work the main objects under scrutiny are *polytopes*. Polytopes, in all guises, constitute a rich and, all in all, intuitive class of geometric objects. Known prior to antiquity, they still furnish vast and active areas of research. From the viewpoint of discrete geometry the important quality is that polytopes admit an interesting study in purely combinatorial terms.

This chapter serves as a reference for basic definitions, results, and notation. We will further assume that the reader has already encountered polytopes and, maybe, some of their combinatorial properties. Readers feeling the urge to acquire more knowledge about polytopes we advise to have a look at the works of Grünbaum (2003) and Ziegler (1995). These are, without doubt, the main sources for polytope theory giving a classical and modern treatment, respectively. For a quick look-up we point the reader to Henk et al. (2004).

One independent notational issue that will accompany us throughout this work is $[n] := \{1, 2, \ldots, n\}$, the set of all natural numbers up to $n \in \mathbb{N}$.

## 1.1 Polytopes

This section presumes basic knowledge of affine geometry.

**Definition 1.1 (Polytope).** A non-empty set $P \subset \mathbb{R}^d$ is called a *polytope* if

($\mathcal{V}$) there is a finite set of points $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$ such that

$$P = \mathsf{conv}(V) := \{\lambda_1 v_1 + \cdots + \lambda_n v_n : \lambda_i \geq 0, \lambda_1 + \cdots + \lambda_n = 1\} \subset \mathbb{R}^d$$

or, equivalently,

($\mathcal{H}$) there are row vectors $a_1, a_2, \ldots, a_m \in \mathbb{R}^d$ and scalars $b_1, b_2, \ldots, b_m \in \mathbb{R}$ such that
$$P = \{x \in \mathbb{R}^d : a_i\, x \leq b_i \text{ for all } i \in [m]\}$$
and the right hand side is a bounded set.

A polytope given by the first part of the definition is called a *$\mathcal{V}$-polytope* or is said to be an *interior representation*, as it describes the polytope as the convex hull of a finite point set. An element $v_i \in V$ is called a *vertex* of $P$ if $P \neq \mathsf{conv}(V \setminus \{v_i\})$ and the set of vertices is denoted by $\mathsf{vert}(P) \subseteq V$. Without loss of generality we will always assume that $\mathsf{vert}(P) = V$ since we can iteratively test the points and remove them from $V$ if necessary. The dimension of a polytope $P$ is $\mathsf{dim}\, P := \mathsf{dim}\,\mathsf{aff}\, P$, the dimension of its affine hull. If $\mathsf{aff}\, P = \mathbb{R}^d$, we call $P$ a *full-dimensional* polytope or, simply, a *$d$-polytope*.

The latter characterization is called an *$\mathcal{H}$-polytope* or *exterior representation*. The reason for that will become clear in a moment, when we introduce the notion of a face. A more economical notation for an $\mathcal{H}$-polytope is given by
$$P = P(A, b) := \{x \in \mathbb{R}^d : A\, x \leq b\}$$
where $A \in \mathbb{R}^{m \times d}$ is a matrix with rows $a_1, \ldots, a_m$ and $b = (b_1, b_2, \ldots, b_m)^T \in \mathbb{R}^m$ is a column vector. For $i \in [m]$ let $A_{\setminus i}$ denote the submatrix of $A$ with the $i$-th row deleted and let $b_{\setminus i}$ be defined likewise. An inequality $a_i\, x \leq b_i$ is called *facet defining* if $P \neq P(A_{\setminus i}, b_{\setminus i})$. As before, we will assume that every row of $(A, b)$ is a facet defining inequality.

The more systematic approach to polytopes is to state the definitions of $\mathcal{V}$- and $\mathcal{H}$-polytope separately and deduce the equivalence from the Main Theorem of polytope theory. For brevity, we cut this route short but the reader will find an excellent exposition in Ziegler (1995). However, the reader might see from that remark that the requirement of $P(A, b)$ being bounded is absolutely necessary for the equivalence. Let us mention that in the unbounded case $P = P(A, b)$ is called an *$\mathcal{H}$-polyhedron*. We will not encounter those in this work.

For $\mathbb{0} \neq c \in \mathbb{R}^d$ and $c_{d+1} \in \mathbb{R}$ we define
$$H(c, c_{d+1}) := \{x \in \mathbb{R}^d : c^T x = c_{d+1}\}$$
to be the (affine) hyperplane determined by $c$ and $c_{d+1}$ and we denote by $H^-(c, c_{d+1}) := \{x : c^T x \leq c_{d+1}\}$ and $H^+(c, c_{d+1})$ respectively the associated (closed) halfspaces.

**Definition 1.2.** Let $P \subset \mathbb{R}^d$ be a polytope and $H = H(c, c_{d+1})$ a hyperplane. We call $H$ a *supporting hyperplane* if $P$ is fully contained in $H^+$ or $H^-$ and $H \cap P \neq \emptyset$. In the affirmative case, we call the intersection $F = P \cap H(c, c_{d+1})$ a *face* of $P$.

In addition to the above definition we agree that $P$ is a face of itself and we call $\emptyset$ the *empty face* of $P$. These, somewhat artificial, faces are called *improper* whereas all other faces are called *proper*. What is apparent from the definition is that every face of $P$ is again a polytope and we can therefore speak of the dimension of a face. The faces of dimension $0, 1, d-2$ and $d-1$ of a $d$-polytope are called *vertices, edges, ridges* and *facets*, respectively. By convention, the empty face has dimension $\operatorname{dim} \emptyset = -1$. Furthermore, if $F$ is a face of $P$ then $V' := \operatorname{vert}(F) \subseteq \operatorname{vert}(P)$ and thus $F = \operatorname{conv}(V')$. From that we see that a supporting hyperplane is equivalently characterized by the property that all vertices lie in one halfspace and facets arise from supporting hyperplanes that are spanned by inclusion-maximal subsets of the vertices.

Let $P = \operatorname{conv} V \subset \mathbb{R}^d$ be a polytope and let $x \in \mathbb{R}^d \backslash P$ be an arbitrary point outside $P$. A face $F$ of $P$ is *visible* from $x$ if for every $y \in F$ the closed line segment $\operatorname{conv}\{x, y\}$ intersects $P$ in $y$. Equivalently, this is the case iff there is a defining hyperplane of $F$ that separates $x$ and $P$. Note that if $P$ is not full dimensional, then $P$ is visible from $x$ if $x \notin \operatorname{aff} P$. For a vertex $v \in V = \operatorname{vert}(P)$ we define $\operatorname{visible}(v\,; P)$ as the set of faces of $P' = \operatorname{conv}(V \backslash \{v\})$ that are visible from $v$.

Another concept in connection with faces that seems less intuitive at first is that of a coface.

**Definition 1.3 (Coface).** Let $P = \operatorname{conv} V$ be a polytope with vertex set $V$. The set $V' \subseteq V$ is called a *coface* if $\operatorname{conv}(V \backslash V')$ is a face of $P$.

According to Grünbaum (2003), the notion of a coface was coined by Micha Perles in connection with Gale transforms (cf. Section 2.1). Let $H = H(c, c_{d+1})$ be a supporting hyperplane and $F = H \cap P$ the induced face. Suppose further that $P \subset H^+$, then all vertices $v \in V$ satisfy $c^T v - c_{d+1} \geq 0$, with equality iff $v \in \operatorname{vert}(F)$. Thus, the coface corresponding to $F$ is $V' = \{v \in V : c^T v - c_{d+1} > 0\}$. We will explore this thinking a little further in the section on Gale transforms.

The combinatorial study of polytopes mostly abstracts from their metric realizations and investigates the facial structure. To be more precise, the set of all faces $\mathcal{L}(P)$ of a polytope $P$ is naturally endowed with a partial order, namely the inclusion relation. This turns $(\mathcal{L}(P), \subseteq)$ into a partially ordered set, called the *face lattice* of $P$, which is a purely combinatorial object. Bearing that in mind, we will write "$F \leq P$" to denote a face $F$ of $P$.

The face lattice of a polytope therefore determines its *combinatorial type* and we call two polytopes $P$ and $P'$ *combinatorially equivalent* if they have isomorphic face lattices.

The number of faces of each dimension is certainly a combinatorial invariant, i.e. combinatorially non-isomorphic polytopes will disagree in these numbers. For a $d$-polytope $P$, this statistic is recorded by the *$f$-vector* $f(P) = (f_0, f_1, \ldots, f_{d-1}, f_d)$ where

$$f_i := \#\{F \leq P : \mathsf{dim}\, F = i\}$$

for $0 \leq i \leq d$.

## 1.2 Simple, Simplicial and neighborly polytopes

The $d$-dimensional *simplex* $\Delta_d \subset \mathbb{R}^d$ is the convex hull of any set of $d+1$ affinely independent points $v_0, v_1, \ldots, v_d \in \mathbb{R}^d$. As the points are free from affine relations, they are indeed the vertices of a polytope and since they affinely span $\mathbb{R}^d$ this polytope is full dimensional. The faces of $\Delta_d$ correspond to all possible subsets of the vertices.

A $d$-polytope $P$ is called a *simplicial* polytope if all of its proper faces are simplices or, equivalently, if its facets are $(d-1)$-simplices. Simplicial polytopes have the property that their vertices can be slightly perturbed without changing the combinatorial type of the convex hull. So it is possible to bring the vertices of a simplicial polytope in general position, a quality of a set of points that we will now define.

**Definition 1.4 (General position).** Let $V \subset \mathbb{R}^d$ be a set of $n \geq d+1$ points. The points $V$ are in *general position* if every (affine) hyperplane contains at most $d$ points.

An important property of "being in general position" is that this property is stable under small perturbations. The next proposition substantiates this statement and even gives an idea of what "small" is supposed to mean.

**Proposition 1.5.** *Let $v_1, \ldots, v_n \in \mathbb{R}^d$ be a finite set of points in general position. Then there is a $\delta = \delta(v_1, v_2, \ldots, v_n) > 0$ such that for each choice $\eta_1, \eta_2, \ldots, \eta_n \in \mathring{B}_\delta = \{x \in \mathbb{R}^d : \|x\| < \delta\}$ the points $v_1+\eta_1, v_2+\eta_2, \ldots, v_n+\eta_n$ are still in general position.*

*Sketch of Proof.* An equivalent definition of points being in general position is that for each $(d+1)$-subset $\{i_0, \ldots, i_d\} \subset [n]$

$$\det \begin{pmatrix} v_{i_0} & \ldots & v_{i_d} \\ 1 & \ldots & 1 \end{pmatrix} \neq 0.$$

Define for every such subset $1 \leq i_0 < i_1 < \cdots < i_d \leq n$ the function

$$P_{i_0,\ldots,i_d}(\eta_{i_0}, \ldots, \eta_{i_d}) := \det \begin{pmatrix} v_{i_0} + \eta_{i_0} & \ldots & v_{i_d} + \eta_{i_d} \\ 1 & \ldots & 1 \end{pmatrix}$$

and with all of them

$$P(\eta_1, \ldots, \eta_n) := \prod_{1 \leq i_0 < i_1 < \cdots < i_d \leq n} P_{i_0,\ldots,i_d}(\eta_{i_0}, \ldots, \eta_{i_d}).$$

$P(\eta_1, \ldots, \eta_n)$ is a multivariate polynomial, hence continuous, in $n \cdot d$ variables and it is zero iff the points $v_1 + \eta_1, v_2 + \eta_2, \ldots, v_n + \eta_n$ are not in general position. By continuity $P^{-1}(\{0\})$ is a closed set and we can thus determine a suitable $\delta$. $\square$

An operation that certainly sustains general position is that of removing a point from a set of $n \geq d+2$ points. For polytopes with vertices in general position that means that every subpolytope, i.e. the convex hull of a spanning subset of the vertices, is again a simplicial polytope and thus, all visible faces are thus a simplices.

Another class of polytopes which are, in a precise sense, dual to simplicial polytopes is the class of simple polytopes. A $d$-polytope is called *simple*, if every vertex is contained in exactly $d$ facets. A property which we will often exploit in subsequent chapters is that simple polytopes are stable under slight perturbations of their facet hyperplanes.

Yet another class of polytope that we will make use of is that of neighborly polytopes. They form an important family of polytopes and one of the foremost members are the cyclic polytopes, to which we will now devote some space and time. But first things first.

**Definition 1.6 (Neighborly polytopes).** Let $P$ be a $d$-polytope, then $P$ is a *$k$-neighborly* polytope, if every subset of $k$ vertices defines a face of $P$. $P$ is called a *neighborly* polytope if it is $\lfloor \frac{d}{2} \rfloor$-neighborly.

It is known (see for example Ziegler (1995), Exercise 0.10) that if a $d$-polytope $P$ is $k$-neighborly with $k > \lfloor \frac{d}{2} \rfloor$ then $P$ is a $d$-simplex. On this note, $k = \lfloor \frac{d}{2} \rfloor$ is the highest degree of neighborliness and it is therefore justified to plainly call polytopes achieving this bound neighborly.

Consider the *moment curve* $\gamma_d : \mathbb{R} \to \mathbb{R}^d$ with $\gamma_d(t) = (t, t^2, \dots, t^d)^T$ and the polytope $C_d(t_1, t_2, \dots, t_n) := \mathsf{conv}\{\gamma_d(t_i) : i \in [n]\} \subset \mathbb{R}^d$ for values $t_1 < t_2 < \cdots < t_n$. It can be shown that $C_d(t_1, t_2, \dots, t_n)$ is a neighborly, simplicial $d$-polytope with $n$ vertices in general position. What is even more amazing is that the combinatorial type is independent of the choices of $t_1, \dots, t_n$ (c.f. Ziegler (1995)). We therefore define $C_d(n) := C_d(t_1, t_2, \dots, t_n)$, for arbitrary values $t_1 < \cdots < t_n$, and call it the $d$-dimensional *cyclic polytope* on $n$ vertices. An additional feature that makes cyclic polytopes so amiable is that its facial structure can be described in purely combinatorial terms.

Before we come to that, let us propose a combinatorial *model* in which to phrase the combinatorics of $C_d(n)$. For a simplicial polytope it suffices to know the vertex sets of facets. Since every proper face is a simplex, the combinatorial structure is then already determined. We will thus solely record the $d$-subsets of vertices corresponding to facets. After choosing an order on the set of vertices, we can encode the vertex-facet incidences as vectors over $\{0,1\}^n$ where $n$ denotes the number of vertices. In detail: let $P$ be a simplicial $d$-polytope with vertices $V = \{v_1, v_2, \dots, v_n\}$ and $F \leq P$ a face. Then we capture the incidences by $\alpha \in \{0,1\}^n$ with

$$\alpha_i = \begin{cases} 0, \text{ if } v_i \in \mathsf{vert}(F) \\ 1, \text{ if } v_i \notin \mathsf{vert}(F) \end{cases}$$

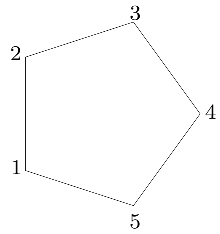
Viewing $\alpha$ as a characteristic vector over $V$ in the usual sense, the reader will find that $\alpha$ denotes a coface. In particular, if $\alpha$ denotes a facet of $P$, then $\alpha$ has $d$ zero entries. For convenience, we define $\mathsf{supp}\,\alpha := \{i \in [n] : \alpha_i \neq 0\}$.

**Theorem 1.7 (Gale's Evenness Condition, Gale (1963)).** *Let $C_d(n)$ be the cyclic $d$-polytope with vertices indexed by $[n] = \{1, 2, \dots, n\}$ in the order in which they occur on the moment curve. Further, let $\alpha \in \{0,1\}^n$ with $d$ zero entries. Then $\alpha$ denotes a facet of $C_d(n)$ if, and only if, for all $i, j \in \mathsf{supp}\,\alpha$ with $i < j$*

$$\#\{k \in [n] : i < k < j, \alpha_k = 0\} \text{ is even.}$$

The reader might consider the following example ridiculous but we will take up this very example in subsequent chapters and we will promise it to be more interesting by then.

| 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| 0 | 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 | 0 |




Anyway, the reader will have no problems with verifying that the pentagon $C_2(5)$ obeys to Gale's Evenness Condition.

## 1.3 Products

One operation that produces new polytopes from old ones is that of taking products. The basic idea is that the pointwise, Cartesian product of two polytopes living in $d$- and $e$-space respectively gives a polytope in $(d+e)$-dimensional space. Noteworthy is that products are a purely combinatorial construction, by which we mean that the facial structure of the product is determined solely by the combinatorics of the factors. We will exploit this fact when we come to deformed products later.

**Definition 1.8 (Product).** Let $P \subset \mathbb{R}^d$ be a $d$-polytope and $Q \subset \mathbb{R}^e$ an $e$-polytope. Then the *product* of $P$ and $Q$ is the Cartesian product

$$P \times Q := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{d+e} : x \in P, y \in Q \right\}.$$

From the above definition it is not at all clear that the product of two polytopes is again a polytope. The next proposition will establish that. Products are among the most basic constructions in polytope theory. We therefore feel at ease with just stating the main properties of products without proof. Details can be found in the afore mentioned literature.

**Proposition 1.9.** *Let* $P = \mathsf{conv}\{p_1, p_2, \ldots, p_n\} = P(A, a) \subset \mathbb{R}^d$ *and* $Q = \mathsf{conv}\{q_1, q_2, \ldots, q_m\} = P(B, b) \subset \mathbb{R}^e$ *be two polytopes and* $P \times Q$ *their product. Then*

*i)* $P \times Q = \mathsf{conv}\{\binom{p_i}{q_j} : i \in [n], j \in [m]\}$

*ii) The points of* $P \times Q$ *are given by the solutions of the system of (facet defining) inequalities*

$$\begin{pmatrix} A & \\ & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \le \begin{pmatrix} a \\ b \end{pmatrix}$$

*where we omitted the zero entries.*

*iii)* *The dimension of the product satisfies* $\mathsf{dim}(P \times Q) = \mathsf{dim}\, P + \mathsf{dim}\, Q$.

*iv)* *The non-empty faces of* $P \times Q$ *are the products of non-empty faces of the factors. On the converse, every product* $F \times G$ *of non-empty faces* $F \leq P$ *and* $G \leq Q$ *is a face of* $P \times Q$.

*v)* *If* $P$ *and* $Q$ *are simple polytopes, then so is* $P \times Q$.

$\square$

The key property for a combinatorial description of $P \times Q$ is point iv) of the above proposition and, combined with iii) proves the following corollary.

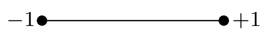
**Corollary 1.10.** *Let* $P$ *and* $Q$ *be polytopes and* $f(P)$, $f(Q)$ *their respective* $f$*-vectors. Then*

$$f_i(P \times Q) = \sum_{\ell=0}^{i} f_\ell(P) f_{i-\ell}(Q)$$

*with the convention that* $f_i = 0$ *if* $i$ *exceeds the dimension of the corresponding polytope.*

Now we illustrate products by a well-known family, the (combinatorial) $d$-cubes.

Let $I = [-1, 1] = \{t \in \mathbb{R} : -1 \leq t \leq 1\}$ be the unit interval. In terms of polytopes, $I$ is a 1-dimensional, simple polytope as depicted in the figure below.




We can identify the non-empty faces of $I$ with the set $\{-, 0, +\}$ in the obvious way: we denote by $-$ and $+$ the left and right vertices of $I$ and let 0 stand for the whole polytope as an improper face.

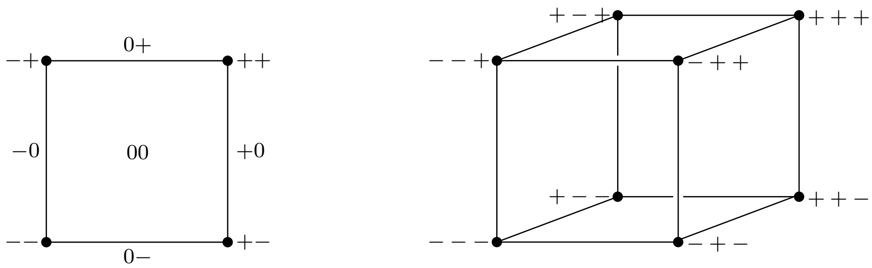
**Definition 1.11 (Standard cube / combinatorial cube).** Let $d \geq 1$. We define $C_d := I^d = I \times I \times \cdots \times I$, the $d$-fold product of $I$, to be the $d$-dimensional *standard cube*. Further, let $P$ be a $d$-polytope, then we call $P$ a *combinatorial* $d$*-cube* if $P$ is combinatorially equivalent to $C_d$.

Cubes constitute a nice class of polytopes which have a simple combinatorial description given as follows.

**Proposition 1.12.** *Let* $C_d$ *be a* $d$*-cube. Then the set of non-empty faces can be identified with the elements* $\alpha \in \{-, 0, +\}^d$. *Furthermore,* $\alpha$ *denotes a* $k$*-face iff* $k = d - \#\mathsf{supp}(\alpha) = \#\{i \in [d] : \alpha_i = 0\}$.

*Proof.* By proposition 1.9 the faces of $I^d$ are products of faces of $I$. We described $I$ in terms of $\{-,0,+\}$. $\alpha \in \{-,0,+\}^d$ specifies faces in all the factors and thus a face of the product. The dimension of the face denoted by $\alpha$ is $1 \cdot k + 0 \cdot (d-k) = k$. $\square$

Figure 1.1 show a two and three dimensional cube together with a (partial) labeling of their faces.




(a) A 2-dimensional cube with labeled non-empty faces.

(b) A 3-cube with labeled vertices.

Figure 1.1: A two and three dimensional cube.

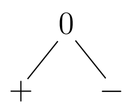
Concerning the $f$-vector, we have the following (trivial) result.

**Corollary 1.13.** *Let $C$ be a combinatorial $d$-cube. Then*

$$f_k(C) = 2^{d-k}\binom{d}{k}$$

*for $0 \le k \le d$. Equivalently, $f_k$ is the coefficient of $t^k$ in the expansion of $f(t) = (2+t)^d$.*

Although we will not make use of it, let us mention that if we equip the set $\{-,0,+\}$ by the an order relation $\preceq$ given by




and extend it componentwise to $\{-,0,+\}^d$ then the partially ordered set $(\{-,0,+\}^d, \preceq)$ is isomorphic to the face lattice of $C_d$ with the minimal element removed.

An explicit $\mathcal{H}$-description of a standard cube is given by

$$C_d = \{x \in \mathbb{R}^d : -1 \leq x_i \leq 1 \text{ for } i \in [n]\}.$$

In subsequent chapters we will be concerned with changing the exterior description of the standard cube such that the combinatorial type does not change. To avoid complicated constructions and an unnecessary formal apparatus, we will exemplify the changes in the inequality system of the cube. In order to make the changes traceable for the reader, here is the inequality system of the standard cube.

$$C_d : \left( \begin{array}{cccccc||c} \pm 1 & & & & & & 1 \\ & \pm 1 & & & & & 1 \\ & & \ddots & & & & \vdots \\ & & & \pm 1 & & & 1 \\ & & & & \pm 1 & & 1 \\ & & & & & \pm 1 & 1 \end{array} \right). \tag{1.1}$$

This is a quite economical description of the cube and thus we think a few explanations are appropriate. The above system consists of $2d$ inequalities. Each row in (1.1) represents two inequalities handling one coordinate at a time. For example, the first row reads as $\pm x_1 \leq 1$ and thus represents $-1 \leq x_1 \leq 1$. As usual, we will represent zero entries by blanks.

# Chapter 2

# Gale Transforms and Subdivisions

'and what is the use of a book,' thought Alice, 'without pictures or conversations?'
—*Lewis Caroll*, from "Alice's Adventures in Wonderland."

Gale transforms and subdivisions play a central rôle in our construction that we will present in the next chapter. Here we will give an introduction (or review, depending on the reader) of both concepts. Gale transforms and especially subdivisions have received enough attention to fill voluminous books (e.g. De Loera, Rambau, and Santos (2005)), so we have to refrain from giving both subjects the treatment they deserve.

We begin by developing the theory of Gale transforms for polytopes, which is far from being the most general setting, but it meets our needs. We will review known and rather unknown facts about Gale transforms, thereby focusing on what will turn out to be useful later. We then proceed by introducing(?) the reader to subdivisions, or more specifically to triangulations of polytopes, again placing emphasis on qualities important for our construction. We conclude the chapter by elaborating on interconnections of Gale transforms and subdivisions, i.e. we will show how to encode information about regular subdivisions into Gale transforms and, vice versa, how perturbed Gale transforms give rise to subdivisions of their underlying polytope.

To simplify the exposition, the general assumption for this chapter is that whenever we are dealing with a set of points $V$ in some $\mathbb{R}^d$, we assume, unless stated otherwise, that the points are in **general** and **convex** position and thus $\mathsf{conv}(V)$ is a simplicial polytope.

A small remark before we really plunge into the subject. After completing this chapter we learned the bitter lesson that some of our main results presented here are far from being new. On the contrary, an exposition treating similar ideas can be found in Lee (1991). Unnecessary to say that we should have spend more time investigating the literature and so instead we intone, once again, that this work is authentic and developed in total unawareness of the aforesaid article.

## 2.1 Gale Transforms

In his seminal book “Convex Polytopes”, Branko Grünbaum (2003) writes

> “The reader will find it well worth his while to become familiar with the concepts of Gale-transforms and Gale-diagrams, since for many of the results obtained through them no alternative proofs have been found so far. It is very likely that the method will yield many additional results.”

and, indeed, we will add to its applicability in the next chapter. In this section we will take up Grünbaums suggestion and give a *familiarizing* exposition.

Gale transforms (and diagrams) are named after David Gale (see Gale (1956)) but they were fully developed by Micha Perles as is documented in Grünbaum (2003). As sources for further study and/or reference we mention Matoušek (2002) for an elementary treatment, Ziegler (1995) for an introduction in connection with oriented matroids, and McMullen (1979) for an algebraically flavored treatise.

The intriguing thing about Gale transforms is that they are almost trivial to define but rather mind-boggling to use. To give the reader a foretaste, let $\mathcal{L} \subseteq \mathbb{R}^n$ be a linear subspace of some $\mathbb{R}^n$ given as the span of a set of vectors. Then basic linear algebra tells us that $\mathcal{L}$ is uniquely determined by $\mathcal{L}^\perp \subset \mathbb{R}^n$, the linear subspace orthogonal to $\mathcal{L}$. Since $\mathcal{L}^\perp$ is linear, it has a basis that is unique up to linear transformations and, by the same argumentation, is uniquely determined by $(\mathcal{L}^\perp)^\perp$, which happens to coincide with $\mathcal{L}$. So far, nothing really spectacular happened.

But, in the study of point configurations, such as the vertices of a polytope, seemingly natural combinatorial data derived from the configuration are the spatial relations of the points to oriented affine hyperplanes, that is for a given hyperplane one can record for every point whether it is on the hyperplane or in which of the induced (open) halfspaces. For example, a face of a polytope is given by its intersection with a supporting hyperplane, i.e. a hyperplane having all vertices either on it or on one *side*. The notion of a Gale transform grows out of the interplay of this combinatorial data and the linear algebra sketched above. The combinatorial data can be captured as a linear space associated to the point configuration and, by linear algebra, this gives rise to an orthogonal or dual linear space which encodes the combinatorial data of another (dual) point configuration. Still with us? We said it is mind-boggling, didn't we?

For the rest of this section, we agree on the following notation. We will consider ordered subsets $V = \{v_1, v_2, \ldots, v_n\}$ of some $\mathbb{R}^d$. Mark, that we do not require all the elements $v_i$ to be distinct and we will distinguish them by their index (thereby making the set notation meaningful). Sometimes it will be convenient to view these sets as matrices and we will use $V = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^{d\times n}$ and the set notation above interchangeably without prior warning. For $I \subseteq [n] = \{1, 2, \ldots, n\}$ we denote by $V_I = \{v_i : i \in I\} \subseteq V$ the induced subset/matrix.

As customary in affine geometry, we pass from affine point configurations to (linear) vector configurations by means of homogenization which will be denoted by

$$V^{\mathsf{hog}} := \begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{(d+1)\times n}.$$

In the homogeneous domain two points $p$ and $q$ are considered equal iff $p = \lambda\, q$ for some $\lambda \neq 0$. So scaling a point such that its last coordinate is equal to 1 is just a way of choosing representatives from an equivalence class. For points with last coordinate equal to zero, this is not possible and they are said to lie in the hyperplane at *infinity*. So linear combinations of homogenized points correspond to *affine* or *convex combinations* of the original points, depending on whether the coefficients are arbitrary or non-negative respectively. For further matters see Berger (1994) or any other (affine) geometry book at hand.

To tell sets of vectors from sets of (affine) points, we will denote the latter with $V^{\mathsf{hog}}$, thus handling affine point configurations in $d$-space as vector configurations in $r = d + 1$ dimensional space. In order to sidestep special cases, we will assume that the set in question linearly spans the ambient space. For $V^{\mathsf{hog}} \subset \mathbb{R}^r$ this means that the points $V$ affinely span $\mathbb{R}^d$.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a function given by $\varphi : x \mapsto c^T x + c_{d+1}$, then we call $\varphi$ an *affine* function or a *linear* function if $c_{d+1} = 0$. On an ordered set $V$ an affine function $\varphi$ gives rise to an *affine value vector* $\varphi(V) \in \mathbb{R}^n$ by recording the value of $\varphi(v_i)$ for each point $v_i \in V$. So the affine value vector for $V$ and $\varphi$ is given by

$$\varphi(V) = c^T V + c_{d+1} \mathbb{1}^T = (c^T, c_{d+1}) V^{\mathsf{hog}}.$$

Considering the last equality, it is evident that an affine function on $V$ is just a linear function on $V^{\mathsf{hog}}$ and therefrom it follows that $\mathsf{Val}(V) \subseteq \mathbb{R}^n$ as well as $\mathsf{Val}(V^{\mathsf{hog}}) \subset \mathbb{R}^n$, the set of linear and affine value vectors, are linear subspaces and, even more, the set of rows of $V$ and $V^{\mathsf{hog}}$ are bases for the corresponding spaces.

Another linear space that we will take into consideration is

$$\mathsf{Dep}(V^{\mathsf{hog}}) := \{\alpha \in \mathbb{R}^n : V\alpha = \mathbb{0}, \mathbb{1}^T \alpha = 0\} = \ker V^{\mathsf{hog}}$$

which happens to be the set of affine dependencies of $V$. This linear subspace is orthogonal to $\mathsf{Val}(V^{\mathsf{hog}})$ which the following proposition assures of.

**Proposition 2.1.** *Let* $\varphi \in \mathsf{Val}(V^{\mathsf{hog}})$ *and* $\alpha \in \mathsf{Dep}(V^{\mathsf{hog}})$. *Then*

$$\langle \varphi, \alpha \rangle = \sum_{i \in [n]} \varphi_i \alpha_i = 0.$$

*Proof.* By construction every $\alpha \in \mathsf{Dep}(V)$ is orthogonal to the rows of the matrix $V$ and thus to every linear combination $\varphi = c^T V \in \mathsf{Val}(V)$. □

Counting dimensions gives $\dim \mathsf{Val}(V^{\mathsf{hog}}) = r$, due to the fact that $V^{\mathsf{hog}}$ has full row rank, and $\dim \mathsf{Dep}(V^{\mathsf{hog}}) = n - \mathsf{rank} V^{\mathsf{hog}} = n - r = n - (d+1)$.

Now we are set to define Gale transforms. Let $V \in \mathbb{R}^{d \times n}$ be a point configuration, $\mathsf{Dep}(V^{\mathsf{hog}}) \subset \mathbb{R}^n$ the space of affine dependencies, and let $G \in \mathbb{R}^{(n-d-1)\times n}$ be a matrix whose rows form a basis for $\mathsf{Dep}(V^{\mathsf{hog}})$. Now comes the major mental leap: we can read $G$ as an ordered set of column vectors and we define $G = \{g_1, g_2, \ldots, g_n\} \subset \mathbb{R}^{n-d-1}$ to be a *Gale transform* of the affine point configuration $V$. The reader might have noticed that there is a certain freedom of choice involved, namely the choice of the basis $G$ for $\mathsf{Dep}(V^{\mathsf{hog}})$. But in what is about to come it will become apparent that any basis will do the job and so we advise the reader to pick his favorite one. However, the one thing we emphasize is that the ordered sets $V$ and $G$ stand in natural bijection to each other, by mapping $v_i \mapsto g_i$ for all $i \in [n]$.

We are ultimately interested in Gale transforms of polytopes, or more precisely, of the vertices of polytopes. Historically, this is the setting in which Gale transforms came into being and, according to Grünbaum (2003), in which Micha Perles coined the notion of a coface. Let $V \backslash V_I$ be the set of vertices of a face $F \leq P$, then there is an affine function $\varphi(x)$ such that $\varphi(v) \geq 0$ for all $v \in V$ and equality is achieved only by the vertices of $F$. Thus the corresponding affine value vector $\varphi(V)$ is non-negative and its support $\{i \in [n] : \varphi(V)_i = \varphi(v_i) > 0\} = I$ determines the coface.

Now comes the reason why Gale transforms are worth studying.

**Theorem 2.2.** *Let $P = \mathsf{conv}\, V \subset \mathbb{R}^d$ be a $d$-polytope with vertices $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$, $G = \{g_1, g_2, \ldots, g_n\} \subset \mathbb{R}^{n-d-1}$ a Gale transform of $V$ and let $I \subseteq [n]$. Then $V_I$ is a coface of $P$ if, and only if, $0 \in \mathsf{relint}\,\mathsf{conv}(G_I)$, i.e. the vectors $G_I$ have a strictly positive dependence.*

*Proof.* Let $\varphi \in \mathsf{Val}(V^{\mathsf{hog}})$ be an affine value vector induced by the face $\mathsf{conv}(V \backslash V_I)$. By definition, $\varphi_i > 0 \Leftrightarrow v_i \in V_I$ and we can assume that $\sum_i \varphi_i = 1$. By proposition 2.1, we have $\sum_{i \in [n]} \varphi_i g_i = \sum_{i \in I} \varphi_i g_i = \mathbb{0}$.

For the converse, note that every positive dependence is a linear combination of $V^{\mathsf{hog}}$ and therefore an affine value vector from which a coface can be read off. $\square$

So questions concerning faces of a polytope $P$ can be posed as questions about positive dependences in the vector configuration $G$. In general, analyzing $G$ instead of $P$ is by no means easier, but the reason for the success of Gale transforms is the reduction in dimension that sometimes happens in the passage from $P$ to $G$. A $d$-polytope having $n \geq d+1$ vertices gives rise to a Gale transform in $(n-d-1)$-dimensional space which is manageable for $n$ small enough. See Ziegler (1995) for examples of *high* dimensional polytopes constructed via their *low* dimensional Gale transforms.

We will benefit from Gale transforms in a totally different way and, in particular, we will derive properties of a Gale transform from the knowledge of the underlying polytope. For these situations we need a characterization of vector configurations that qualify as Gale transforms.

**Proposition 2.3.** *Let $G = \{g_1, g_2, \ldots, g_n\} \subset \mathbb{R}^k$ be a set of vectors satisfying $G\,\mathbb{1} = \sum_i g_i = \mathbb{0}$. Then $G$ is a Gale transform of an $(n-k-1)$-dimensional polytope if, and only if, for every linear hyperplane both induced open half-spaces contain at least two of the vectors of $G$.*

*Proof.* Let $V = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^{(n-k-1)\times n}$ be a basis of $\mathsf{Dep}(G^{\mathsf{hog}})$. Now the points $v_1, v_2, \ldots, v_n$ are the vertices of a polytope if, and only if, no $v_i$ is affinely dependent on $V\backslash v_i$. This is the case if, and only if, every affine dependence has at least two positive and two negative coefficients. Since the affine dependences of $V$ are affine value vectors coming from linear hyperplanes on $G$, this completes the proof. □

Another important issue is the question of how the dimension of a face $F = \mathsf{conv}(V\backslash V_I)$ relates to the (linear) dimension of $G_I$. For a polytope whose vertices are in general position, the answer is rather simple.

**Proposition 2.4.** *Let $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ with $n \geq d+1$ be a set of points and $G = \{g_1, \ldots, g_n\} \subset \mathbb{R}^{n-d-1}$ its Gale transform. Then the points of $V$ are in general position if, and only if, no linear hyperplane contains more than $n-d-2$ vectors of $G$ (the vectors of $G$ are in general position).*

*Proof.* Yet another, equivalent characterization of the points $V$ being in general position is that every inclusion-minimal affine dependence involves $d+2$ points. So let $\alpha = c^T G \in \mathsf{Dep}(V^{\mathsf{hog}})$ be an affine dependence with minimal support, i.e. $\alpha$ has at least $d+2$ non-zero entries. By construction, $\alpha$ is a linear combination of the rows of $G$ and thus a (linear) value vector $\alpha \in val(G)$ with at most $n-(d+2)$ zero entries. That means that the hyperplane $\{x : c^T x = 0\}$ contains at most $n-d-2$ vectors of $G$. □

So in case of a polytope with vertices in general position we get the following corollary.

**Corollary 2.5.** *Let $V \subset \mathbb{R}^d$ be the vertices of a polytope $P$ in general position. Then $V_I \subseteq V$ is a coface of $P$ if and only if the vectors $G_I$ positively span $\mathbb{R}^{n-d-1}$.*

And for the general position case we can choose a basis for $\mathsf{Dep}(V^{\mathsf{hog}})$ having a particularly nice form.

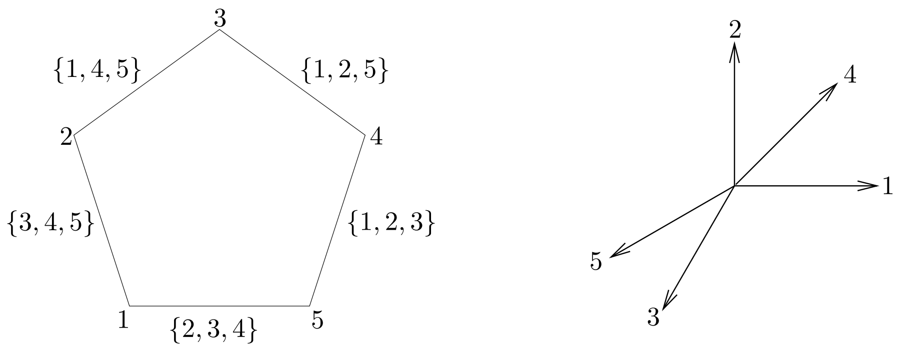
**Proposition 2.6.** *Let $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ be a set of points in general position with $n \geq d+1$. Then $V$ has a Gale transform $G \in \mathbb{R}^{(n-d-1)\times n}$ of the form $G = (\mathtt{I}_{n-d-1} G')$ with $G' \in \mathbb{R}^{(n-d-1)\times(d+1)}$.*

*Proof.* Let $k := n-d-1$ and $V' = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$. For each $i = 1, \ldots, k$ the set $\{v_i\} \cup V'$ is minimally affinely dependent and thus has an affine dependence of the form $(1, g_{i,k+1}, g_{i,k+2}, \ldots, g_{i,n})$. Then the matrix

$$G = \begin{pmatrix} 1 & & & & g_{1,k+1} & g_{1,k+2} & \cdots & g_{1,n} \\ & 1 & & & g_{2,k+1} & g_{2,k+2} & \cdots & g_{2,n} \\ & & \ddots & & \vdots & & & \\ & & & 1 & g_{k,k+1} & g_{k,k+2} & \cdots & g_{k,n} \end{pmatrix}$$

is the desired Gale transform. □

We close with an example of a polytope, namely a $C_2(5)$, and its Gale transform.




(a) A pentagon with its facets labeled by the cofaces, that is the vertices not contained in the facet.

(b) Gale transform of the pentagon to the left. The labels correspond to the vertices.

Figure 2.1: The figure shows a pentagon and its Gale transform. The vertices of the pentagon are in general position, which is reflected in the Gale transform.

## 2.2 Subdivisions

Subdivisions are a means of *subdividing* a geometrical object into "smaller", possibly more manageable objects/parts. They are used in diverse and often seemingly unrelated areas of mathematics ranging, for instance, from algebraic topology (from where they originated) to the theory of binary trees (cf. Rambau (2000)). Here we will study them because subdivisions of certain polytopes (surprisingly) *carry* the combinatorics of the projected deformed polytopes we will construct later.

The section is organized as follows: we start with the definition of polyhedral complexes which naturally lead to subdivisions. We then introduce the reader to regular subdivisions as well as to methods to obtain them and go into the subtleties of realizing certain regular subdivisions, or lexicographic subdivisions to be more precise, geometrically.

For further particulars we refer the reader to De Loera et al. (which is in preparation at the time of writing) as well as to Rambau (2000) on which this section is based. For a clear and brief introduction see also Lee (2004).

Broadly speaking subdivisions are polytopal complexes associated to a (finite) set of points in some $\mathbb{R}^d$.

**Definition 2.7 (Polytopal Complex).** A non-empty set of polytopes $\mathcal{S}$ in some $\mathbb{R}^d$ is a *polytopal complex* if it satisfies

i) if $F \leq P \in \mathcal{S}$ then $F \in \mathcal{S}$, and (*Closure property*)

ii) $P, P' \in \mathcal{S}$ then $P \cap P' \leq P$ and $P \cap P' \leq P'$. (*Intersection property*)

The *dimension* of $\mathcal{S}$ is $\mathsf{dim}(\mathcal{S}) := \mathsf{max}\{\mathsf{dim}(P) : P \in \mathcal{S}\}$ and $\mathcal{S}$ is called *pure* if all its inclusion-maximal polytopes have the same dimension. The *underlying set* (or *polyhedron*) of $\mathcal{S}$ is the underlying point set $\|\mathcal{S}\| = \bigcup_{P \in \mathcal{S}} P$ and its *vertices* are $\mathsf{vert}(\mathcal{S}) = \bigcup_{P \in \mathcal{S}} \mathsf{vert}(P)$.

So a polytopal complex is a set of polytopes closed under taking faces and whose polytopes lie face-to-face. The polytopes in $\mathcal{S}$ are called *faces* or *cells*. We will stick to the latter term to avoid confusion with faces of polytopes. If all cells in $\mathcal{S}$ are simplices then $\mathcal{S}$ is usually called a *simplicial complex*. The polytopal complexes that we will consider in here are all *finite*, which means that they contain only finitely many polytopes.

To illustrate the definition let $P$ be a polytope. Then two associated pure polytopal complexes are the *complex of the polytope* $\mathcal{C}(P) = \{F : F \leq P\}$ and its *boundary complex* $\mathcal{C}(\partial P) = \{F : F < P\} = \mathcal{C}(P)\backslash\{P\}$. If $P$ is a simplicial polytope then $\mathcal{C}(\partial P)$ is a simplicial complex. If $P$ is a simplex then $\mathcal{C}(P)$ obviously simplicial.

Suppose we have a finite set of polytopes $\mathcal{P} = \{P_1, P_2, \ldots, P_k\}$ satisfying condition i) of the above definition. We can turn $\mathcal{P}$ into a polytopal complex by adding the faces of each $P_i$. For that we define the *closure* of such a set as $\mathsf{cl}(\mathcal{P}) := \bigcup_{P \in \mathcal{P}} \mathcal{C}(P)$.

**Example 2.8.** Let $\mathcal{S}$ be a polytopal complex and $v \in \mathsf{vert}(\mathcal{S})$ a vertex.

i) The closure of the set of faces of $\mathcal{S}$ that contain $v$ is a polytopal complex called the *(closed) star* of $v$ and denoted by $\mathsf{star}(v; \mathcal{S}) := \mathsf{cl}\{F \in \mathcal{S} : v \in F\}$.

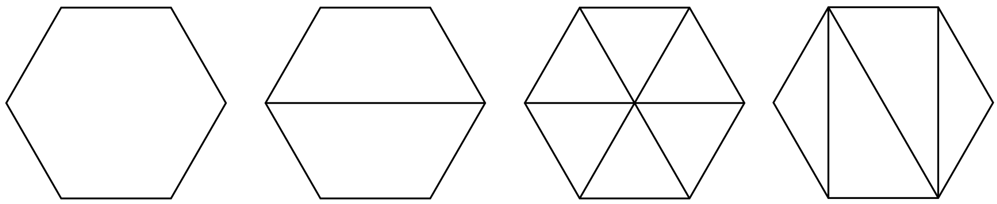
ii) Dually, if we consider the set of faces of $\mathcal{S}$ not containing $v$ then this again gives us a polytopal complex from which we can recover $\mathsf{star}(v; \mathcal{S})$ in $\mathcal{S}$ and vice versa. This complex is called the *anti-star* of $v$ and is given by $\mathsf{astar}(v; \mathcal{S}) := \{F \in \mathcal{S} : v \notin F\}$. Note that no closure is necessary: If a polytope does not contain $v$ then every face of it does neither.

iii) Star and anti-star of a vertex cover the whole complex. Their intersection is the polytopal complex $\mathsf{link}(v; \mathcal{S}) := \mathsf{star}(v; \mathcal{S}) \cap \mathsf{astar}(v; \mathcal{S})$ called the *link* of $v$.

These three operations produce new, albeit smaller, complexes from a given one. They can, however, be stated more generally by, for example, considering (anti-)stars of higher dimensional faces (cf. Grünbaum (2003)).

**Definition 2.9 (Subdivision).** Let $V \subset \mathbb{R}^d$ be a set of points. A *subdivision* of $V$ is a (pure) polytopal complex $\mathcal{S}$ such that $\|\mathcal{S}\| = \mathsf{conv}(V)$. It is called a subdivision *without new vertices* if $\mathsf{vert}(\mathcal{S}) \subseteq V$. If $\mathcal{S}$ is a simplicial complex then $\mathcal{S}$ is called a *triangulation* of $V$.

An illustration of the definition is given in Figure 2.2.



(a) Trivial subdivision

(b) Non-simplicial Subdivision

(c) Triangulation with new vertex

(d) Lexicographic Triangulation

Figure 2.2: Different subdivisions of a hexagon.

One class of subdivisions that is, in a sense, particularly well behaved is that of regular subdivisions. A regular subdivision can be thought of as a subdivision of a point set induced by a projection of a higher dimensional polytope. In order to give a satisfying definition of regular subdivisions we have to introduce the notion of a lower face.

**Definition 2.10 (Lower face).** Let $P \subset \mathbb{R}^{d+1}$ be a $(d+1)$-polytope and $F < P$ a face. Then $F$ is a *lower face* if $x - \lambda e_{d+1} \notin P$ for all $x \in F$ and $\lambda > 0$.

Equivalently, a face $F$ is a lower face if there is a defining hyperplane $H(c, \delta)$ of $F$, i.e. $F = H(c, \delta) \cap P$, whose outer normal $c = (c', c_{d+1})^T \in \mathbb{R}^{d+1}$ satisfies $c_{d+1} < 0$. We denote by $\mathcal{F}^\ell(P)$ the set of all lower faces of $P$ and call it the *lower envelope* of $P$.

**Definition 2.11 (Regular subdivision).** Let $V \subset \mathbb{R}^d$ be a set of points and $\mathcal{S}$ a subdivision of $V$. Then $\mathcal{S}$ is a *regular subdivision* if there is a polytope $P_{\mathcal{S}} \subset \mathbb{R}^{d+1}$ satisfying

i) the projection of $P_{\mathcal{S}}$ obtained by deleting the last coordinate yields $\mathsf{conv}(V)$, and

ii) the set of lower faces of $P_{\mathcal{S}}$ yields $\mathcal{S}$, under the projection $\mathbb{R}^{d+1} \to \mathbb{R}^d$ that deletes the last coordinate. In particular, $\mathcal{F}^\ell(P_{\mathcal{S}}) \cong \mathcal{S}$.

We will solely be interested in subdivisions without new vertices and so the above definition can be rephrased in the following way: if $\mathcal{S}$ is a subdivision of $V = \{v_1, v_2, \dots, v_n\}$ without new vertices then $\mathcal{S}$ is a regular subdivision if there are heights $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ such that the lower envelope of

$$V^w := \mathsf{conv}\begin{pmatrix} v_1 & v_2 & \cdots & v_n \\ w_1 & w_2 & \dots & w_n \end{pmatrix} \subset \mathbb{R}^{d+1}$$

is isomorphic to $\mathcal{S}$. On the other hand, every height vector $w \in \mathbb{R}^n$ defines a regular subdivision of $V$ without new vertices which we will denote by $\mathcal{T}(V, w) := \mathcal{F}^\ell(V^w)$.

There are two rather common operations to obtain regular subdivisions which are called pulling and pushing a vertex.

**Definition 2.12 (Pulling for polytopes).** Let $P$ be a $d$-polytope and $v \in \mathsf{vert}(P)$ a vertex. Then the result of *pulling* $v$ is the subdivision $\mathsf{pull}(v; P)$ of $P$ given by

$$\begin{aligned} \mathsf{pull}(v;P) :&= \{v * F : F \text{ is a face of } P \text{ not containing } v\} \\ &= \{v * F : F \in \mathsf{astar}(v; \mathcal{C}(P))\}. \end{aligned}$$

**Definition 2.13 (Pushing for polytopes).** Let $P = \mathsf{conv}(V)$ be a $d$-polytope and $v \in V$ a vertex. The subdivision obtained by pushing $v$ is

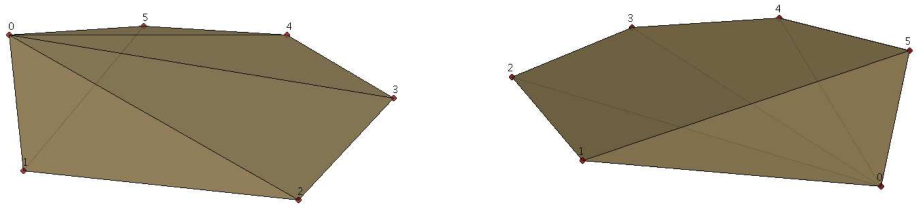
$$\mathsf{push}(v;P) := \{v * F : F \in \mathsf{visible}(v\,; P)\} \cup \mathcal{C}(\mathsf{conv}(V \backslash v)).$$

In case $P$ is a pyramid with apex $v$ we get $\mathsf{push}(v;P) := \mathcal{C}(P)$.

Note that $\mathsf{pull}(v;P)$ and $\mathsf{push}(v;P)$ leave any simplex, or more generally a pyramid (with apex $v$), unchanged. Figure 2.3 shows a pulling and a pushing subdivision of a hexagon.

If $\mathcal{S}$ is a subdivision of $V$ then we can push/pull a vertex $v$ in $\mathcal{S}$ by

$$\begin{aligned} \mathsf{pull}(v;\mathcal{S}) &:= \mathsf{astar}(v;\mathcal{S}) \cup \{\mathsf{pull}(v;F) : F \in \mathsf{star}(v;\mathcal{S})\}, \text{ and} \\ \mathsf{push}(v;\mathcal{S}) &:= \mathsf{astar}(v;\mathcal{S}) \cup \{\mathsf{push}(v;F) : F \in \mathsf{star}(v;\mathcal{S})\}. \end{aligned}$$



(a) Pulling vertex 0 gives a triangulation of the hexagon.

(b) Pushing vertex 0 gives a subdivision with one cell being a pentagon.

Figure 2.3: Illustration of the operations $\mathsf{pull}(v; P)$ and $\mathsf{push}(v; P)$ with $P$ being a hexagon.

Applying push/pull to a subdivision gives a new (possibly unchanged) subdivision of the same set of points. So after choosing an ordering of the points $V = \{v_1, v_2, \ldots, v_n\}$ we can specify for each point whether we pull or push it. This idea is condensed in the definition of a lexicographic subdivision.

**Definition 2.14 (Lexicographic Subdivision).** Let $V = \{v_1, v_2, \ldots, v_n\}$ be a full-dimensional set of points and $s_1, s_2, \ldots, s_k \in \{-, +\}$ with $k \leq n$. Then the *lexicographic subdivision*

$$\mathsf{Lex}(V; s_1, \ldots, s_k) := \mathsf{Lex}(\mathcal{C}(\mathsf{conv}\, V); s_1, \ldots, s_k)$$

of $V$ is defined recursively by

$$\mathsf{Lex}(\mathcal{S}; s_i, s_{i+1}, \ldots, s_k) := \begin{cases} \mathsf{Lex}(\mathsf{push}(v_i; \mathcal{S}); s_{i+1}, \ldots, s_k), & \text{if } s_i = + \\ \mathsf{Lex}(\mathsf{pull}(v_i; \mathcal{S}); s_{i+1}, \ldots, s_k), & \text{if } s_i = - \end{cases}$$

$$\mathsf{Lex}(\mathcal{S}; ) := \mathcal{S}$$

Lexicographic subdivisions were studied by Sturmfels (1991) in connection with Gröbner bases of toric varieties.

In general this definition is redundant in the sense that different pulling/pushing sequences lead to the same subdivision. For the general position case, this fact is specified by the next proposition.

**Proposition 2.15.** *Let $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$ be the vertices of a $d$-polytope in general position and $s_1, s_2, \ldots, s_k \in \{-, +\}$ with $k \leq n$ such that there is at least one component with a negative sign and $p := \mathsf{min}\{i \in [k] : s_i = -\}$. Then* $\mathsf{Lex}(V; s_1, s_2, \ldots, s_k) \cong \mathsf{Lex}(V; s_1, s_2, \ldots, s_p)$

*Proof.* Let $\mathcal{S}' = \mathsf{Lex}(V; s_1, \ldots, s_{p-1})$. All cells of $\mathcal{S}'$ but one are simplices and the remaining cell is isomorphic to $Q = \mathsf{conv}\{v_p, v_{p+1}, \ldots, v_n\}$. This is due to the fact that the cells emerging by a push on a set of vertices in general position are simplices (cf. Definition 2.13). Now the next step in the course of constructing $\mathsf{Lex}(V; s_1, s_2, \ldots, s_k)$ replaces $Q$ by a pulling triangulation with respect to $v_p$. Thus $\mathsf{Lex}(V; s_1, s_2, \ldots, s_p)$ is a triangulation of $V$ and refinements by means of pushing or pulling leave it invariant. $\square$

So we can define $\mathsf{Lex}_p(V) := \mathsf{Lex}(V; s_1, s_2, \ldots, s_p)$ $p \leq n$ with $s_i = +$ for $i \in [p-1]$ and $s_p = -$. If $p = n$ then the triangulation arises by pushing all vertices in the order in which they occur.

Before we plunge into further matters concerning lexicographic subdivisions, we intermit to have a detailed look at lexicographic subdivisions of cyclic polytopes.

Let $C_d(n) = \mathsf{conv}\{\gamma_d(1), \gamma_d(2), \ldots, \gamma_d(n)\}$ be *the* cyclic $d$-polytope on $n$ vertices. We identify the vertices with the set $[n] = \{1, 2, \ldots, n\}$ in its natural ordering. By Theorem 1.7 the facial structure is given by cofaces represented by vectors $\alpha \in \{0,1\}^n$ that satisfy the Gale's Evenness Condition. If we delete the vertex $i$ we obtain, surprise, the cyclic polytope $C_d(n-1)$. Let $\alpha \in \{0,1\}^{n-1}$ be a coface of $C_d(n-1)$ and denote by $\alpha[i \leftarrow 1] = (\alpha_1, \ldots, \alpha_{i-1}, 1, \alpha_i, \ldots, \alpha_{n-1})$ the extended vector with 1 inserted at position $i$. Then $\alpha[i \leftarrow 1]$ might or might not satisfy Gale's Condition. In the case it does, then $\alpha[i \leftarrow 1]$ is a coface of $C_d(n)$ that does not contain vertex $i$. Otherwise, $\alpha$ is a non-face as it is covered by $i$. This observation enables us to state the following.

**Proposition 2.16 (Pushing/Pulling for cyclic polytopes).** *Let $n \geq d+2$ and $\alpha \in \{0,1\}^{n-1}$ be a coface of $C_d(n-1)$ and $i \in [n]$. Then $\alpha[i \leftarrow 0]$ is a cell of $\mathsf{pull}(i; C_d(n))$ if, and only if, $\alpha[i \leftarrow 1]$ satisfies Gale's Evenness Condition. Otherwise, $\alpha[i \leftarrow 1]$ is a cell of $\mathsf{push}(i; C_d(n))$.* $\square$

If we push/pull the vertices in the given, natural order, we get

**Corollary 2.17.** *Let $\alpha$ be a cofacet of $C_d(n-1)$. Then $\alpha[i \leftarrow 0]$ is a cell of $\mathsf{pull}(1; C_d(n))$ if $\alpha$ starts with an* even *number of zeros. Otherwise, it corresponds to a cell of $\mathsf{push}(1; C_d(n))$.*

Trivially, every facet $\alpha$ of $C_d(n-1)$ either starts with an even or odd number of zeros. The following table displays all lexicographic subdivisions $\mathsf{Lex}_p(C_2(5))$.

| p | $s_1$ $s_2$ | 1 2 3 4 5 |
|---|---|---|
| 1 | $-$ $\pm$ | 0 0 0 1 1 |
| | | 0 1 0 0 1 |
| | | 0 1 1 0 0 |
| 2 | $+$ $-$ | 0 0 1 1 0 |
| | | 1 0 0 0 1 |
| | | 1 0 1 0 0 |
| 3 | $+$ $+$ | 0 0 1 1 0 |
| | | 1 0 0 0 1 |
| | | 1 0 1 0 0 |

We introduced pushing-/pulling-subdivisions, and hence lexicographic subdivisions, as regular subdivisions. For that to be true, we have to verify that there are heights realizing the subdivision. We now state (and of course prove) one of our main theorems of this section. It asserts that under rather mild restrictions on the heights the signs of the heights indeed determine a lexicographic subdivision.

**Theorem 2.18.** *Let $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$ be a set in convex and general position and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ a non-zero height vector satisfying $|w_{i+1}| \leq \varepsilon |w_i|$ for all $i \in [n-1]$. Further, let $p := \mathsf{min}(\{i : w_i < 0\} \cup \{n-d\})$. Then for sufficiently small $\varepsilon > 0$ the subdivision $\mathcal{T}(V, w)$ induced by $w$ is isomorphic to $\mathsf{Lex}_p(V)$.*

Before we prove the theorem we need a rather technical lemma.

**Lemma 2.19.** *Let $V = \{v_1, v_2, \ldots, v_n\} \subset \mathbb{R}^d$ be as above and $w = (w_1, w_2, \ldots, w_n) \in \mathbb{R}^n$ a non-zero height vector satisfying $|w_i| \leq \varepsilon |w_1|$ for $i = 2, \ldots, n$. Then for sufficiently small $\varepsilon > 0$ $\mathsf{star}(v_1; V^w) \cong \{v_1 * F : F \in \mathcal{C}(\partial P')\}$ with $P' = \mathsf{conv}(V \backslash \{v_1\})$.*

We postpone the proof of the lemma till later (thereby making it easier for the reader to skip it).

*Proof of Theorem 2.18.* In the proof we denote by $V_i := \{v_i, v_{i+1}, \ldots, v_n\}$ a subset of the vertices and by $V_i^w$ the convex hull of the lifted subset (so $V_i$ is the projection of $V_i^w$ along the last coordinate). We proceed by induction on $|V_i|$ starting with $V_p$.

The heights of $V_p^w$ satisfy $|w_j| \leq \varepsilon |w_p|$ for $j > p$ and so by Lemma 2.19 the lower faces of $V_p^w$ form a subdivision of $\mathsf{conv}\, V_p$ as obtained by pulling $v_p$. If $p = n - d$ then $V_p$ is a simplex which is unaffected by pulling, otherwise $w_p$ is negative and the star of $v_p$ in $V_p^w$ is a pyramid over $\mathsf{conv}\, V_{p+1}$. So the lower faces involving $v_p$ are joins over the boundary of $\mathsf{conv}\, V_{p+1}$.

Now assume that for $i+1 \leq p$ the lower faces of $V_{i+1}^w$ give a subdivision of $V_{i+1}$ isomorphic to pushing $v_i$ to $v_{p-1}$ and pulling $v_p$. If we add $v_i$ then none of the lower faces of $V_{i-1}$ vanish due to Lemma 2.19. New faces involve only the shadow boundary of $V_{i+1}$ and $w_i > w_{i+1}$. $\square$

*Proof of Lemma 2.19.* We show that the vertex figure of $v_1$ in $V^w$ is isomorphic to $P'$. We do that by considering the intersection of the cone $C = \{v_1^{w_1} + t(v_1^{w_1} - x) : x \in V^w\}$ (with cone point $v_1^w$) with the hyperplane $H = \{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\}$.

Let $\hat{v}_i$ be the point of intersection of $\mathsf{aff}\{v_1^{w_1}, v_i^{w_i}\}$ with $H$ for $i = 2, \ldots, n$. Then $\hat{v}_i$ is given by

$$\begin{aligned} \hat{v}_i &= v_1 + \lambda_i(v_i - v_1) \text{ with } \lambda_i \text{ satisfying} \\ 0 &= w_1 + \lambda_i(w_i - w_1) \\ \Leftrightarrow \lambda_i &= \frac{w_1}{w_1 - w_i}. \end{aligned}$$

Moreover, $\lambda_i > 0$ since $w_1$ and $w_1 - w_i$ have the same sign and so every ray emanating from $v_1^{w_1}$ through a point of $V^w$ intersects $H$ in a unique point.

As the points $V$ are in general position, Proposition 1.5 assures that every $\Delta < \delta(v_1, \ldots, v_n)$ is the radius of an open ball centered at $v_i$ in which we may perturb $v_i$ while still retaining general position. Thus the points $\hat{v}_i$ are still in general position if $\|\hat{v}_i - v_i\| < \Delta$ for all $i = 2, \ldots, n$ (cf. Figure 2.4). In terms of $\varepsilon$ that means

$$\begin{aligned} \|\hat{v}_i - v_i\| &= \|(v_1 - v_i) - \lambda_i(v_1 - v_i)\| \\ &= \|(1 - \lambda_i)(v_1 - v_i)\| \\ &= \frac{|w_i|}{|w_1 - w_i|} \|v_1 - v_i\| \\ &< \frac{\varepsilon |w_1|}{|w_1| - \varepsilon |w_1|} \|v_1 - v_i\| \\ &= \frac{\varepsilon}{1 - \varepsilon} \|v_1 - v_i\| < \Delta \end{aligned}$$

which is satisfied for $\varepsilon < \frac{\Delta}{D+\Delta}$ with $D := \mathsf{max}\{\|v_i - v_1\| : i = 2, \ldots, n\}$. So $\mathsf{conv}\{\hat{v}_2, \ldots, \hat{v}_n\} \cong \mathsf{conv}\{v_2, \ldots, v_n\}$ is isomorphic to the vertex figure $V^w/v_1$ which proves our claim. $\square$

So far, we have treated lexicographic subdivisions and *perturbed*, or rather extended, Gale transforms in much detail. Concluding this chapter, we will put the pieces together by showing how to encode lexicographic subdivisions in Gale transforms.

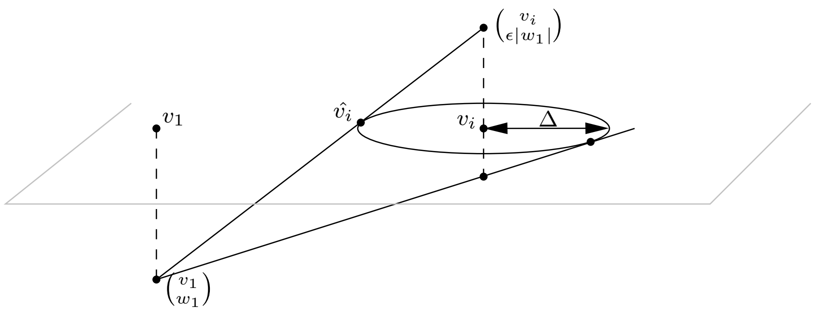



Figure 2.4: Illustration of the proof of Lemma 2.19

**Theorem 2.20.** *Let $V = \{v_1, \dots, v_n\} \subset \mathbb{R}^d$ be the vertices of a $d$-polytope $P$ in general position and*

$$w' = (w, \mathbb{0})^T = (w_1, w_2, \dots, w_{n-d-1}, 0, \dots, 0)^T \in \mathbb{R}^V \cong \mathbb{R}^n$$

*a height vector. Further, let $G = (\mathtt{I}_{n-d-1}\, G') \in \mathbb{R}^{(n-d-1)\times n}$ be a Gale transform of $V$. Then the extension of $G$ by one column*

$$G_w := (-w\, G) = (-w\ \mathtt{I}_{n-d-1}\, G')$$

*is a Gale transform of a polytope $P_w \subset \mathbb{R}^{d+1}$ realizing the regular subdivision of $V$ induced by $w$, i.e. $\mathcal{F}^\ell(P_w) \cong \mathcal{T}(V; w)$. The remaining faces are $v_0 * \mathcal{C}(\partial P)$, where $v_0$ is the vertex that corresponds to the first column of $G_w$.*

*Proof.* It is rather obvious that $G_w$ is a Gale transform of a polytope. The necessary and sufficient conditions for $G_w$ being a Gale transform are that 1) $G_w$ has full rank and 2) that for each oriented linear hyperplane at least two points of $G_w$ (viewed as a set of column vectors) lie in the positive halfspace. Since $G$ already satisfies both conditions so does $G_w$.

Next we will determine the vertices of the polytope $P_w$. For that, observe that $G_w$ arises as the result of column operations on the matrix $\hat{G} = (\mathbb{0}\, G)$. So there is a non-singular matrix $U \in \mathbb{R}^{(n+1)\times(n+1)}$ with $k := n - d - 1$ such that $G_w U = \hat{G}$ and

$$U := \begin{pmatrix} 1 & & & & & & \\ w_1 & 1 & & & & & \\ \vdots & & \ddots & & & & \\ w_k & & & 1 & & & \\ 0 & & & & 1 & & \\ \vdots & & & & & \ddots & \\ 0 & & & & & & 1 \end{pmatrix}$$

Now, $\hat{G}$ is a Gale transform of $\hat{P} = v_0 * P$, a pyramid with base $P$ and apex $v_0$. Taking the pyramid over a polytope is a combinatorial construction and the only condition is that $v_0$ does not lie in the affine hull of $P$. If we elevate our setting into the realms of projective geometry by introducing homogeneous coordinates, the points

$$\hat{V} := \begin{pmatrix} \mathbb{0} & v_1 & v_2 & \cdots & v_n \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \end{pmatrix} \in \mathbb{R}^{(d+2)\times(n+1)}$$

serve our needs as vertices of $\hat{P}$. Readers familiar with projective geometry will notice that we took the liberty of choosing a point at infinity as our apex $v_0$.

So, before dehomogenizing, the actual vertices of the polytope $P_w$ are given by

$$V_w := \hat{V}U^T = \begin{pmatrix} \mathbb{0} & v_1 & v_2 & \cdots & v_k & v_{k+1} & \cdots & v_n \\ 1 & w_1 & w_2 & \cdots & w_k & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{pmatrix}$$

as is easily verified. Applying a suitable projective transformation to $V_w$ gives the desired result. $\square$

**Definition 2.21.** For a simplicial $d$-polytope $P = \mathsf{conv}\, V \subset \mathbb{R}^d$ with vertices $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ in general position and a height vector $w \in \mathbb{R}^{n-d-1}$ satisfying the conditions of Theorem 2.18 and $p \geq 0$ accordingly, let $G_w$ be the Gale transform of Theorem 2.20. We define $\mathsf{Lex\text{-}Pyr}_p(P) \subset \mathbb{R}^{d+1}$ as the polytope corresponding to $G_w$. So $\mathsf{Lex\text{-}Pyr}_p(P)$ has as proper faces the cells of the lexicographic triangulation $\mathsf{Lex}_p(P)$ and $v_0 * \mathcal{C}(\partial P)$, where $v_0$ is the apex as in Theorem 2.20.

From the theorem we get two ugly (*viz.* technical) yet useful by-products.

**Corollary 2.22.** *Let $V$ and $G$ be as in Theorem 2.20 and $w = (w_1, \ldots, w_{n-d-1})$. If $\|w\|$ sufficiently small, then the* perturbed *Gale transform*

$$\tilde{G}_w = \left(\begin{array}{c|ccccc} w_1 & 1 & & & & \\ & w_2 & 1 & & & \\ & & \ddots & \ddots & & G' \\ & & & w_{n-d-1} & 1 & \end{array}\right) \tag{2.1}$$

*represents a regular subdivision of $V$ with height vector $w' = (w'_1, w'_2, \ldots, w'_{n-d-1})$,*

$$w'_i = (-1)^i \prod_{\ell=1}^{i} w_\ell \text{ for } i \in [n-d-1].$$

*Proof.* For $|w_i|$ sufficiently small the part of the matrix right to the bar is still a Gale transform of a polytope combinatorially equivalent to $\mathsf{conv}(V)$. Since Gale transforms are unique up to non-singular linear transformations we can restore the identity matrix right of the bar by row-operations applied to $\tilde{G}_w$. This linear transformation also acts on the first column resulting in $-w'$ and so the transformed matrix looks like $(-w' \; \mathtt{I}_{n-d-1} \, G'')$. Then using Theorem 2.20 proves the claim. $\square$

In terms of *controlled* perturbations of Gale transforms, the next corollary goes a considerable step further.

**Corollary 2.23.** *For the prerequisites as in Corollary 2.22 let*

$$u = (u_1, 0, u_3, 0, \ldots, 0, u_{k-1}) \in \mathbb{R}^{n-d-1}$$

*be such that* $|u_i| < |w_i| \cdot |w_{i-1}|$ *for* $i = 1, 3, \ldots, k-1$ *and consider the perturbed Gale transform*

$$\tilde{G}_{w,u} = \begin{pmatrix} w_1 & 1 & & & & & & \\ u_1 & w_2 & 1 & & & & & \\ & 0 & w_3 & \ddots & & & & \\ & & u_3 & \ddots & \ddots & & & G' \\ & & & \ddots & \ddots & \ddots & & \\ & & & & 0 & w_{k-1} & 1 & \\ & & & & & u_{k-1} & w_k & 1 \end{pmatrix}.$$

*Then* $\tilde{G}_{w,u}$ *induces the same subdivision as* $\tilde{G}_w$.

*Proof.* With the help of suitable row operations the matrix $\tilde{G}_{w,u}$ can be brought into the form of (2.1). The entries $w_i$ are modified to $w_i' = w_i - \frac{u_i}{w_{i-1}}$ for $i$ even and $w_i' = w_i$ otherwise. Obviously, $w_i$ and $w_i'$ have the same sign and hence induce the same subdivision as $\|w\|$ is sufficiently small. $\square$

# Chapter 3

# Deformed Products and Projections

Im großen Garten der Geometrie kann sich jeder nach seinem Geschmack einen Strauß pflücken.
—*David Hilbert*

We finally reach the peak level in this last chapter. We will retrace the polytopes constructed by Joswig and Ziegler (2000) and Ziegler (2004), but our approach starts from a, say, more conceptual point of view. In both of the mentioned articles, the polytopes under scrutiny are constructed as projections of high dimensional deformed products of polytopes. The key ingredients to both constructions are (well) known facts about projections of polytopes as well as suitable, although ad hoc, deformations of the polytopes to be projected. Due to the latter, a full combinatorial description of the projections was either hard to come by or not available at all. This we will remedy by our treatment of the subject.

Needless to say, that the interest in *Neighborly Cubical Polytopes* of Joswig and Ziegler and the *Deformed Products of Polytopes* of Ziegler goes beyond their *novel* construction techniques. Neighborly cubical polytopes turned out to solve several open problems in polytope theory (cf. the above mentioned articles), and they continue to open up new areas of application, even as we write. We warmly recommend to the reader the recent article of Joswig and Schröder (2005), where neighborly cubical polytopes serve as carriers for the embedding of polyhedral surfaces into 3-space. One more thing worth mentioning is that neighborly cubical polytopes once were the *fattest* polytopes in existence. Fatness is a quantity common to the study of $f$-vector cones. At present[1], the high score in fatness is held by the Projected Deformed Products of Polygons. Given that both families of polytopes draw on similar construction principles, one might presume that *Deformed Products* will rise to further glory ... so stay tuned.

Highly complex structures are easy to provide in high dimensions, but, alas, in *low* dimensions such as $\mathbb{R}^4$ we lack the ingenious imagination for such constructions. So the main reason for studying projections is their ability to carry over some of the structure to lower dimensions. In that spirit, we start off by investigating projections of polytopes, thereby introducing (to the reader) the concept of a *strictly preserved* face. Determining the combinatorial type of a projection a-priori is in general a rather hopeless venture/undertaking. The mentioned articles circumvent these problems by considering *orthogonal products* whose canonical projections are trivial, i.e. the projections yield the involved factors, and alter them in a controlled manner giving rise to fascinating specimens of geometry. These ideas will be studied under the headline of *deformed products*.

## 3.1 Projections

Let $\pi : \mathbb{R}^d \to \mathbb{R}^e$ be a surjective, linear map with $d \geq e$, then we call $\pi$ an *projection map*. As $\pi$ is an epimorphism between vector spaces, it is given by $x \mapsto \pi(x) = Bx$ where $B \in \mathbb{R}^{e \times d}$ is a matrix of full row rank ($= e$).

We take for granted that the projections of polytopes are polytopes as well. But nevertheless, there is no combinatorial approach to the theory of projections, i.e. in general there is no combinatorial data that determines beforehand the type of the projection. For example, the well-known Minkowski sum of two polytopes is a projection of a product under the mapping $(x, y) \mapsto x + y$. It does, however, depend intrinsically on the coordinatization of the involved polytopes (see Ziegler (1995) for details).

In this section we will investigate some properties of projections. In particular, we study faces that retain their structural properties under projection. For that we start with the following definition.

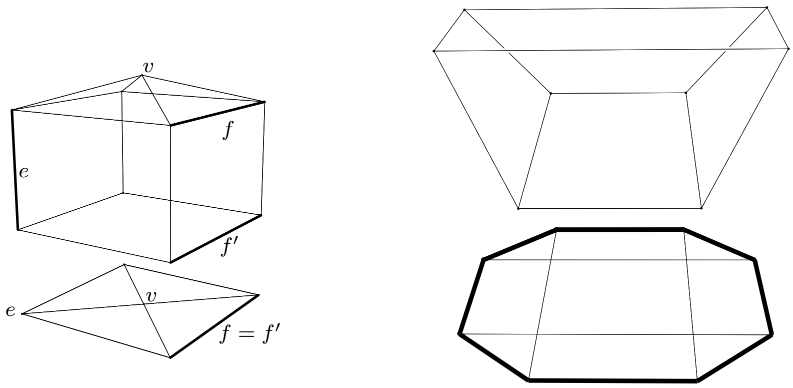
**Definition 3.1 (Strictly preserved faces).** Let $P \subset \mathbb{R}^d$ be a $d$-polytope, $\pi : \mathbb{R}^d \to \mathbb{R}^e$ a projection map and $Q := \pi(P)$ the projection of $P$. A $k$-face $F \leq P$ is *strictly preserved* by $\pi$ if

[1] August 18, 2005

i) $G = \pi(F)$ is a $k$-dimensional face of $Q$ combinatorially equivalent to $F$, and

ii) the preimage $\pi^{-1}(G)$ is $F$.

The first condition is rather intuitive as it states that the combinatorial type of a face is preserved. The second condition demands that there is no other face that *collapses* onto $F$. Figure 3.1 shows (inevitably plane) drawings of two different 3-polytopes as well as their projections to the plane.




(a) Projection of a stacked cube. No non-empty face is strictly preserved.

(b) Projection of a combinatorial cube. All vertices and some of the edges are strictly preserved as indicated

Figure 3.1: In Figure (a) the vertex $v$ falls into the interior, the projection of the vertical edge $e$ is still a face but degenerates to a vertex, and the horizontal edges $f$ and $f'$ get identified by the projection. Figure (b) shows a Goldfarb cube whose vertices are preserved by the projection.

Although the definition expresses formally what one should have in mind when talking about faces being strictly preserved by projection, it hardly gives any idea of how to check the conditions for a face before the actual projection is carried out.

**Proposition 3.2.** *Let $P \subset \mathbb{R}^d$ be a d-polytope, $Q = \pi(P) \subset \mathbb{R}^e$ the image of a projection $\pi$ of $P$ and $F \leq P$ a face of $P$. Then $\pi(F)$ is a face of $Q$ if $F$ has a defining hyperplane $H(c, c_{d+1})$ such that $c$ is in the row span of $B$. Moreover, every face of $Q$ arises in that way.*

*Proof.* Let $(c, c_{d+1}) \in \mathbb{R}^{d+1}$ be such that $F = H(c, c_{d+1}) \cap P$ and suppose that $c$ is in the row span of $B$, i.e. there is a $\hat{c} \in \mathbb{R}^e$ such that $c^T = \hat{c}^T B$. For an arbitrary $x \in P$ let $\hat{x} = \pi(x) \in Q$ be its projection, then $c_{d+1} \geq c^T x = \hat{c}^T Bx = \hat{c}^T \hat{x}$ and with equality only if $x \in F$. So $H(\hat{c}, c_{d+1})$ is a supporting hyperplane of $Q$ and $\pi(F) = H(\hat{c}, c_{d+1}) \cap Q$.

For the second statement, let $G \leq Q$ be an arbitrary face with $G = Q \cap H(\hat{c}, c_{d+1})$. Define $c^T := \hat{c}^T B$ as the pullback with respect to $\pi$. Then it is easily seen that $H(c, c_{d+1})$ is a supporting hyperplane of $P$ with a non-empty intersection with $P$. $\square$

The next proposition tackles the second condition as dictated by Definition 3.1.

**Proposition 3.3.** *Let $P \subset \mathbb{R}^d$ be a $d$-polytope and $\pi : \mathbb{R}^d \to \mathbb{R}^e$ an orthogonal projection. Let $F \leq P$ be a $k$-face of $P$ and $\mathsf{aff}(F) = x + L$ be its affine hull with $x \in F$ and $L$ a linear subspace. Then the image $\pi(F)$ is combinatorially equivalent to $F$ if, and only if, $\mathsf{ker}(\pi) \cap L = \{\mathbb{0}\}$.*

*Proof.* The image is combinatorially equivalent to $F$ if the restriction $\pi|_F$ is a bijection. To simplify matters, we note that $\pi|_F$ is surjective since $\pi$ is and so it suffices to show that $\pi|_L$ is injective. If two points $x, y \in L$ get identified by $\pi$, that is $\pi(x) = \pi(y)$, then $x - y$ lies in $\mathsf{ker}(\pi)$. Thus $\mathsf{ker}(\pi) \cap L = \{\mathbb{0}\}$ if, and only if, $\pi|_L$ is injective. $\square$

The affine hull of a $k$-face $F \leq P$ is given by $\mathsf{aff}(F) = \{x \in \mathbb{R}^d : A'x = b'\}$ where $A' \in \mathbb{R}^{\ell \times d}$ is a matrix the rows of which are normals to facets containing $F$. $\ell \geq n - k$.

Later, we will restrict ourselves to projections along coordinate axes and thus we define the *canonical* projection $\pi_e : \mathbb{R}^d \to \mathbb{R}^e$ that projects to the last $e$ coordinates, i.e. $\pi_e : \mathbb{R}^d \ni (x, x') \mapsto x' \in \mathbb{R}^e$.

For the canonical projection the above results become considerably simpler.

**Corollary 3.4 (Ziegler (2004), Proposition 3.2).** *Let $P \subset \mathbb{R}^d$ be a $d$-polytope and $F$ a $k$-face of $P$. Further, let $A \in \mathbb{R}^{\ell \times (d-e)}$ be the matrix whose $\ell \geq d - k$ rows are the first $d - e$ components of normals to facets containing $F$. Then $F$ is strictly preserved by $\pi_e$ if, and only if, the rows of $A$ positively span $\mathbb{R}^{d-e}$.*

*Proof.* If the rows of $A$ positively span $\mathbb{R}^{d-e}$ they span $\mathbb{R}^{d-e}$ in the ordinary sense and so $A$ has full row rank. Furthermore, there is a $\lambda \in \mathbb{R}^\ell$ with $\lambda > 0$ and $\lambda^T A = \mathbb{0}^T$. Hence $F$ satisfies the conditions of proposition (3.2) and (3.3).

For the converse, observe that $A$ has to have full row rank anyway and $F$ has a normal $c \in \mathbb{R}^d$ as in proposition (3.2) iff $c = (\mathbb{0}, c')$ with $c' \in \mathbb{R}^k$. $\square$

## 3.2 Deformed Products

The term *deformed product* was coined in Amenta and Ziegler (1999) where the authors give a unified approach to several, somewhat classical, polytopes and their construction. These polytopes mainly arose in the study of linear programs and pivot rules of the simplex algorithm but, from a different perspective, these polytopes exhibit extremal properties not unlike those studied in the next section.

We will give their original definition of (rank 1) deformed products as well as some minor results about the exterior representation of deformed products. The emphasis on *rank 1* hints at possible generalizations which we will explore thereafter.

**Definition 3.5 (Deformed Product – rank 1).** Let $P \subset \mathbb{R}^d$ and $V, W \subset \mathbb{R}^e$ be convex polytopes, and let $\varphi : P \to \mathbb{R}$ be an affine functional with $\varphi(P) \subseteq [0,1]$. Then the *(rank 1) deformed product* of $(P, \varphi)$ and $(V, W)$ is

$$(P,\varphi) \bowtie (V,W) := \left\{ \begin{pmatrix} x \\ \varphi(x)v + (1-\varphi(x))w \end{pmatrix} : x \in P, v \in V, w \in W \right\} \subset \mathbb{R}^{d+e}$$

This definition gives a point-by-point view of the subject and thus is unusable for computational efforts. To make deformed products combinatorial assume that $V$ and $W$ are normally equivalent polytopes, i.e. they are combinatorially equivalent polytopes whose corresponding facet normals coincide. Hence $V = P(B, b)$ and $W = P(B, b')$ where the rows of $B \in \mathbb{R}^{n\times e}$ are facet normals and $b, b' \in \mathbb{R}^n$ are right hand sides leading to combinatorially equivalent polytopes. Next, let $P = P(A, a)$ with $A \in \mathbb{R}^{m\times d}$ and let the affine functional be given by $\varphi(x) = c^T x + c_0$ and define $C := (b - b')c^T \in \mathbb{R}^{n\times d}$.

**Proposition 3.6.** *For $V, W$ normally equivalent the deformed product $(P, \varphi) \bowtie (V, W)$ is given by the solutions to the following set of inequalities*

$$\begin{pmatrix} A & \mathbb{0} \\ C & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} a \\ (1-c_0)b + c_0 b' \end{pmatrix}.$$

*Proof.* Due to an observation of R. Seidel (cf. Remark 3.8 in Amenta and Ziegler), the deformed product $(P, \varphi) \bowtie (V, W)$ is given by the projection along $t$ of

$$\left\{ \begin{pmatrix} x \\ u \\ t \end{pmatrix} : \begin{array}{rl} x & \in P \\ u & = (1-t)v + tw, v \in V, w \in W \\ t & \in [0,1] \end{array} \right\} \cap \left\{ \begin{pmatrix} x \\ u \\ t \end{pmatrix} : c^T x + c_0 = t \right\}$$

This polytope is given by

$$\begin{aligned} A\,x \qquad\qquad\qquad &\le a \\ B\,y + (b - b')\,t &\le b \\ t &\le 1 \\ -t &\le 0 \\ c^T x \qquad\qquad -t &= -c_0. \end{aligned}$$

Now, one Fourier-Motzkin elimination step gives the desired result. □

Obviously, the matrix $C$ has rank 1, which should explain the supplement in the name of the definition.

The key point to note from the above proposition is that in case $V$, and therefore $W$, are simple polytopes the condition that $C$ has rank 1 is far too restrictive.

**Definition 3.7 (Deformed Product – rank r).** Let $P = P(A, a) \subset \mathbb{R}^d$ and $Q = P(B, b) \subset \mathbb{R}^e$ two full-dimensional simple polytopes with $B \in \mathbb{R}^{n \times e}$. For a matrix $C \in \mathbb{R}^{n \times d}$ with $\mathsf{rank}\; C = r$, let $M > 0$ such that $P(B, M\,b - C\,x) \cong Q$ for all $x \in P$. We define the *rank r deformed product* $P \bowtie_C Q$ of $P$ and $Q$ to be the polytope whose points satisfy

$$\begin{aligned} A\,x \qquad &\le a \\ C\,x + B\,y &\le M\,b. \end{aligned}$$

Note that such an $M$ always exists: dividing the last (matrix) inequality by $M$, the entries in $\frac{1}{M}C$ become arbitrarily small. Hence the above polytope is equivalent to the standard product $P \times Q$ with a small perturbation of the facet normals of the second factor. But since $P \times Q$ is a simple polytope, small perturbations do not change the combinatorics. This remark proves the following proposition.

**Proposition 3.8.** *The polytopes $P \times Q$ and $P \bowtie_C Q$ are combinatorially equivalent.* □

All polytopes that we will construct are deformed products. We conclude this section with the presentation of a classical family of iterated rank 1 deformed products, the *Goldfarb Cubes.*

**Example 3.9 (Goldfarb Cubes).** In the mid 80's Donald Goldfarb refuted the conjectured polynomial running time of the simplex algorithm with the *shadow boundary* pivot rule. He did so by constructing an infinite family of combinatorial cubes with the property that all vertices lie on the *shadow boundary* which in our terminology reads: all vertices survive the projection to the plane. We will study them here mainly for two reasons. First, they are prime examples of deformed products and their projections can be understood without sophisticated tools. As the constructions get more involved, the way will be paved (or is it already?) with phrases like "for $\varepsilon$ sufficiently small". So the second reason is that for the Goldfarb Cubes all the nebulous parameters can be pinpointed! Let us mention that the following construction deviates from the one given by Donald Goldfarb. But since it has the same qualities as the original we will call it Goldfarb Cubes nevertheless.

**Proposition 3.10.** *For $n \geq 3$ and $0 < \varepsilon < 1$, let $G_n$ be defined by*

$$G_n : \left(\begin{array}{ccccc|cc||c} \pm\varepsilon & & & & & & & 1 \\ 1 & \pm\varepsilon & & & & & & M^1 \\ & 1 & \ddots & & & & & \vdots \\ & & \ddots & \pm\varepsilon & & & & M^{n-3} \\ & & & 1 & & \pm\varepsilon & & M^{n-2} \\ -1 & -1 & \cdots & -1 & & & \pm\varepsilon & M^{n-1} \end{array}\right).$$

*Then $G_d$ is a combinatorial $n$-cube if $M \geq \frac{2}{\varepsilon} > 1$.*

*Proof.* We will verify by induction that for $1 \leq i \leq n$ the possible values of $x_i$ form a proper (non-singular) interval for all valid choices of $x_1, x_2, \ldots, x_{i-1}$, thus proving the claim. Since the distortion in the last inequality is different from the others, we treat it separately.

We prove that any solution of the above system satisfies $M^i > |x_i|$. For $i = 1$ see that $M > \frac{1}{\varepsilon} > |x_1|$. For $i > 1$ we note that, by induction $|x_{i-1}| < M^{i-1}$, and thus

$$|x_i| \leq \frac{1}{\varepsilon}(M^{i-1} + |x_{i-1}|) < \frac{2}{\varepsilon}M^{i-1} \leq M^i.$$

So $(2 \leq i \leq n-1)$: $M^{i-1} - |x_{i-1}| > M^{i-1} - M^{i-1} = 0$. For the last inequality we get that

$$\begin{aligned} M^{n-1} - \sum_{i=1}^{n-2} |x_i| &> M^{n-1} - \sum_{i=1}^{n-2} M^i > M^{n-1} - \frac{M^{n-1} - 1}{M - 1} \\ &= \frac{M^n - 2M^{n-1} + 1}{M - 1} = \frac{(M^{\frac{n}{2}} - 1)^2}{M - 1} > 0. \end{aligned}$$

$\square$

The next thing we have to look into is that the projection to the last two coordinates preserves all vertices. Later, we will see that Chapter 2 furnished us with tools to treat questions about projections but now we will verify it by hand.

To guarantee that a vertex $v \in G_n$ survives the projection it is, by Corollary 3.4, sufficient that the $n-2$ components of normals to $v$ positively span $\mathbb{R}^{n-2}$. So let $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \{-,+\}^n$ denote facets that contain $v$ and let

$$\bar{G}_\sigma := \begin{pmatrix} \sigma_1\varepsilon & & & & \\ 1 & \sigma_2\varepsilon & & & \\ & 1 & \ddots & & \\ & & \ddots & \sigma_{n-2}\varepsilon \\ & & & 1 \\ -1 & -1 & \cdots & -1 \end{pmatrix} \in \mathbb{R}^{n\times(n-2)}$$

be the matrix whose rows are the first $n-2$ components of the corresponding facet normals.

**Proposition 3.11.** *Let $\bar{G}_\sigma$ be as above. The rows of $\bar{G}_\sigma$ positively span $\mathbb{R}^{n-2}$ if $\varepsilon < 1/2$.*

*Proof.* What we show is that the claim is true if we delete the first row from $\bar{G}_\sigma$, and thus for $\bar{G}_\sigma$ as well.

First, $\bar{G}_\sigma$ has full row rank independent of the choice of $\sigma$ and $\varepsilon$: Deleting the first and last row leaves a square matrix with determinant 1.

Now, suppose that we have a dependence given by $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{R}^n$ such that $\alpha_1 = 0$, which amounts for that fact that we dropped the first row from $\bar{G}_\sigma$. It follows that $\alpha_n \neq 0$ and so we can assume that $\alpha_n = 1$. The other coefficients are subject to $\alpha_i = 1 - \sigma_{i-1}\varepsilon\alpha_{i-1}$ for $i = 2, \dots, n-1$, as can be seen by inspecting the $(i-1)$-st column. It follows by induction that $\alpha_i \leq \sum_{k=0}^{i-1} \varepsilon^k = \frac{1-\varepsilon^i}{1-\varepsilon}$ and thus $\alpha_i \geq 1 - \varepsilon\alpha_{i-1} \geq 1 - \varepsilon\frac{1-\varepsilon^i}{1-\varepsilon} > 0$ if $\varepsilon < \frac{1}{2}$. □

See Figure 3.1(b) for a 3-dimensional Goldfarb Cube and its projection.

The Goldfarb Cubes constructed in the last section had the property that a projection to the plane preserved all vertices. In the terminology of Grünbaum (2003) that means that the $d$-cube, for $d \geq 3$, is *dimensionally* 0-*ambiguous*, i.e. there is a polytope $P$ with $\mathsf{dim}(P) < d$ whose 0-skeleton is isomorphic to that of the $d$-cube. As the 0-skeleton refers to the set of vertices, the above statement boils down to the fact that there are $e$-polytopes ($e \neq d$) different from the $d$-cube that have $2^d$ vertices, which should not come as a surprise.

In Joswig and Ziegler (2000) the question about dimensional $k$-ambiguity of $d$-cubes is investigated for higher $k$. The main result in their article is the following.

**Theorem 3.12 (Joswig and Ziegler (2000), Theorem 17).** *Let $n, d \in \mathbb{N}$ with $n \geq d$. Then there is a cubical $d$-polytope whose $(\lfloor \frac{d}{2} \rfloor - 1)$-skeleton is isomorphic to that of a $n$-cube.*

They prove the result by explicitly constructing a combinatorial $n$-cube whose projection to $d$-space satisfies the claim made by the theorem and they give a complete description of the combinatorial structure of the projection, namely a *Cubical Gale's Evenness Condition.*

In this section, we will reinvent the polytopes of Joswig and Ziegler but with considerably more degrees of freedom.

**Construction 3.13.** Let $n, d \in \mathbb{N}$ with $n > d$ and let $Q \subset \mathbb{R}^{d-2}$ be a neighborly simplicial $(d-2)$-polytope on $n-1$ vertices in general position. Furthermore, choose an arbitrary but fixed ordering of the vertices of $Q$ and let $G \in \mathbb{R}^{(n-d)\times(n-1)}$ be a Gale transform of $Q$ of the form $G = (\mathtt{I}_{n-d}\, G')$, where we denote by $g_1^T, g_2^T, \ldots, g_{d-1}^T \in \mathbb{R}^{n-d}$ the columns of $G'$. We define the polytope $\hat{C}_n(Q, d)$ by the following set of inequalities

$$\left(\begin{array}{cccc|cccc||l}
\pm\varepsilon & & & & & & & & b_1\\
1 & \pm\varepsilon & & & & & & & b_2\\
 & 1 & \ddots & & & & & & \vdots\\
 & & \ddots & \pm\varepsilon & & & & & b_{n-d}\\
 & & & 1 & \pm\varepsilon & & & & b_{n-d+1}\\
\multicolumn{4}{c|}{\text{———}\; g_1 \;\text{———}} & & \pm\varepsilon & & & b_{n-d+2}\\
\multicolumn{4}{c|}{\vdots} & & & \ddots & & \vdots\\
\multicolumn{4}{c|}{\text{———}\; g_{d-1} \;\text{———}} & & & & \pm\varepsilon & b_n
\end{array}\right)$$

Again, each row corresponds to two facets. The single vertical bar indicates that we will project the polytope to the last $d$ coordinates.

From the discussion in the last section it should be clear that $\hat{C}_n(Q, d)$ is an instance of a deformed product and for appropriate choices of the $b_i$'s it is indeed a combinatorial $n$-cube.

Define $\mathsf{NCP}_d(Q) := \pi_d(\hat{C}_n(Q, d)) \subset \mathbb{R}^d$ as the projection of $\hat{C}_n(Q, d)$ to the last $d$ coordinates. We contend that $\mathsf{NCP}_d(Q)$ proves the claim made by Theorem 3.12 and thus, we will verify that every $k$-face of $\hat{C}_n(Q, d)$ with $k = \lfloor \frac{d}{2} \rfloor - 1$ survives the projection to $d$-space.

In light of Corollary 3.4, this can be accomplished by showing that for every $k$-face $F \le \hat{C}_n(Q,d)$ the first $n-d$ components of normals to facets containing $F$ positively span $\mathbb{R}^{n-d}$.

We will approach that task in the following, redundant way. We show that for an arbitrary vertex $v \in \mathsf{vert}(\hat{C}_n(Q,d))$ all $k$-faces that contain $v$ survive the projection. By, generally abided, abuse of notation we identify the vertex $v$ with the intersection of its incident facets.
So we choose $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \{-,+\}^n$ and define

$$\bar{G}_\sigma := \begin{pmatrix} \sigma_1\varepsilon & & & & \\ 1 & \sigma_2\varepsilon & & & \\ & 1 & \ddots & & \\ & & \ddots & \sigma_{n-d}\varepsilon & \\ & & & 1 & \\ \hline & & g_1 & & \\ & & \vdots & & \\ \hline & & g_{d-1} & & \end{pmatrix} \in \mathbb{R}^{(n-d)\times n}$$

as the matrix made up of the first $n-d$ components of facet normals that define the vertex $\sigma$. We actually see that $\sigma_1, \dots, \sigma_{n-d}$ alone determine $\bar{G}_\sigma$ but we insist on the definition because it will turn out appropriate when we analyze the facial structure of the projection.

Now here comes the thing: By Corollary 2.22 this matrix can be viewed as a perturbed Gale transform for a pyramid over $Q$ and therefore encodes a regular subdivision of $Q$ with heights $w \in \mathbb{R}^{n-1}$, where

$$w_i = (-1)^i \varepsilon^i \prod_{j=1}^{i} \sigma_i.$$

Stronger yet, the height vector satisfies $|w_{i+1}| \le \varepsilon |w_i|$ and thus, for $\varepsilon > 0$ small enough, meets the condition of Theorem 2.18. Consequently $\bar{G}_\sigma$ corresponds to $\mathsf{Lex\text{-}Pyr}_p(Q)$ where

$$p := \mathsf{min}(\{i \in [n-d] : \sigma_i \ne -\} \cup \{n-d\}). \tag{3.1}$$

**Theorem 3.14.** *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{-,0,+\}^n$ name a face of $\hat{C}_n(Q,d)$. Then $\alpha$ is a face of $\mathsf{NCP}_d(Q)$ if, and only if, $|\alpha| = (|\alpha_1|, \dots, |\alpha_n|) \in \{0,1\}^n$ is a coface of $\mathsf{Lex\text{-}Pyr}_p(Q)$ where $p$ is defined by (3.1) by a vertex contained in $\alpha$.*

*Proof.* Let $\alpha$ be a coface of $\mathsf{Lex\text{-}Pyr}_p(Q)$. In the language of Gale transforms that means that $\hat{G}_\sigma(\alpha)$, the rows of $\hat{G}_\sigma$ corresponding to nonzero entries of $\alpha$, have a strictly positive dependence. The polytope $\mathsf{Lex\text{-}Pyr}_p(Q)$ is a $(d-1)$-dimensional simplicial polytope and so every coface contains $n-(d-1)$ vertices that affinely span $\mathbb{R}^{d-1}$ and, hence, $\hat{G}_\sigma(\alpha)$ has full rank. Corollary 2.22 witnesses the fact that $\alpha$ is a face of $\mathsf{NCP}_d(Q)$.

Conversely, suppose that $\alpha$ is a face of $\mathsf{NCP}_d(Q)$. Then $\alpha$ corresponds to a face of $\hat{C}_n(Q,d)$ having a normal orthogonal to the direction of projection and, by the virtue of Gale transforms, is a coface of $\mathsf{Lex\text{-}Pyr}_p(Q)$. The rank condition is trivially satisfied according to the argumentation above. $\square$

Note, that all facets of $\mathsf{NCP}_d(Q)$ are strictly preserved faces of $\hat{C}_n(Q,d)$ and, hence, $\mathsf{NCP}_d(Q)$ is indeed a cubical polytope.

The upcoming corollary is implicit in the theorem.

**Corollary 3.15.** *The projection $\mathsf{NCP}_d(Q)$ has the $(\lfloor\frac{d}{2}\rfloor-1)$-skeleton of the $n$-cube.*

*Proof.* By our choice, $Q$ is a neighborly $(d-2)$-polytope which means that every subset of $\lfloor\frac{d-2}{2}\rfloor = \lfloor\frac{d}{2}\rfloor - 1$ vertices is a face of $Q$. This property stays intact if we switch to $\mathsf{Lex\text{-}Pyr}_p(Q)$. $\square$

The combinatorial structure is intrinsically dependent on $Q$ whose description can be rather wild. Luckily, cyclic polytopes as well as their lexicographic triangulations are rather straightforward, to which, we hope, the reader agrees after having read Section 2.2. To this end, we now give a thorough account on the combinatorics of the neighborly cubical polytopes for cyclic polytopes.

**Theorem 3.16 (Cubical Gale's Evenness Condition).** *Let $\mathsf{NCP}_d(n) := \mathsf{NCP}_d(Q)$ with $Q = C_{d-2}(n-1)$ be the cyclic $(d-2)$-polytope on $n-1$ vertices in the order induced by the moment curve. For $\alpha \in \{-,0,+\}^n$ with $d-1$ zero entries denote by $p \geq 0$ the least number such that $\alpha_{p+1} = 0$. Then $\alpha$ is a facet of $\mathsf{NCP}_d(n)$ if one the following conditions is satisfied:*

- *$p = 0$ and $|\alpha| = (|\alpha_2|,\dots,|\alpha_n|) \in \{0,1\}^{n-1}$ satisfies the (ordinary) Gale's Evenness Condition.*
- *$0 < p \leq n-d$ and $\alpha$ is of the form $(-,-,\dots,-,\underset{\substack{\uparrow\\ p}}{\sigma},0,\alpha_{p+2},\dots,\alpha_n)$.*

  *Then $(|\alpha_{p+2}|,\dots,|\alpha_n|)$ satisfies Gale's Evenness Condition and*

  - *starts with an **odd** number of zeros, if $\sigma = +$, or*
  - *starts with an **even** number of zeros, if $\sigma = -$.*

- *$p = n - d + 1$ and $\alpha$ is of the form $(-, -, \cdots, -, \pm, 0, 0, \ldots, 0)$.*

*Proof.* Consider the polytope $\mathsf{Lex\text{-}Pyr}_p(Q)$ that carries the combinatorics according to Theorem 3.14. For $p = 0$, $\alpha$ needs to be a co-facet containing the vertex $v_0$. By definition, these are $v_0 * F$ where $F$ is a facet of $Q$ and hence $(|\alpha_2|, \ldots, |\alpha_n|)$ has to satisfy Gale's Evenness Condition.

For $p > 0$, the corresponding co-facet is contained in $\mathsf{Lex}_p(Q)$ whose combinatorial description was given in Corollary 2.17. □

In the next table we list the facets of $\mathsf{NCP}_4(6)$, at which the Cubical Gale's Evenness Condition can be seen.

| $p$ | $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6$ |
|---|---|
| 0 | $0\ 0\ 0\ \pm\ \pm\ \pm$ |
| | $0\ \pm\ 0\ 0\ \pm\ \pm$ |
| | $0\ \pm\ \pm\ 0\ 0\ \pm$ |
| | $0\ \pm\ \pm\ \pm\ 0\ 0$ |
| | $0\ 0\ \pm\ \pm\ \pm\ 0$ |
| 1 | $+\ 0\ 0\ 0\ \pm\ \pm$ |
| | $+\ 0\ \pm\ 0\ 0\ \pm$ |
| | $+\ 0\ \pm\ \pm\ 0\ 0$ |
| | $-\ 0\ 0\ \pm\ \pm\ 0$ |
| 2 | $-\ +\ 0\ 0\ 0\ \pm$ |
| | $-\ +\ 0\ \pm\ 0\ 0$ |
| | $-\ -\ 0\ 0\ \pm\ 0$ |
| 3 | $-\ -\ +\ 0\ 0\ 0$ |
| | $-\ -\ -\ 0\ 0\ 0$ |

We claimed at the beginning of this section that we can add degrees of freedom to the construction of the neighborly cubical polytopes as compared to Joswig and Ziegler (2000). In order to justify our claim we need the following observation.

**Lemma 3.17.** *Let $v$ be a vertex of $\mathsf{NCP}_d(Q)$, determined by $\alpha \in \{-, +\}^n$ . Then the vertex figure $\mathsf{NCP}_d(Q)_{/v}$ is isomorphic to $\mathsf{Lex\text{-}Pyr}_p(Q)$ with $p$ given by (3.1) and, hence, contains $\mathcal{C}(\partial Q)$.*

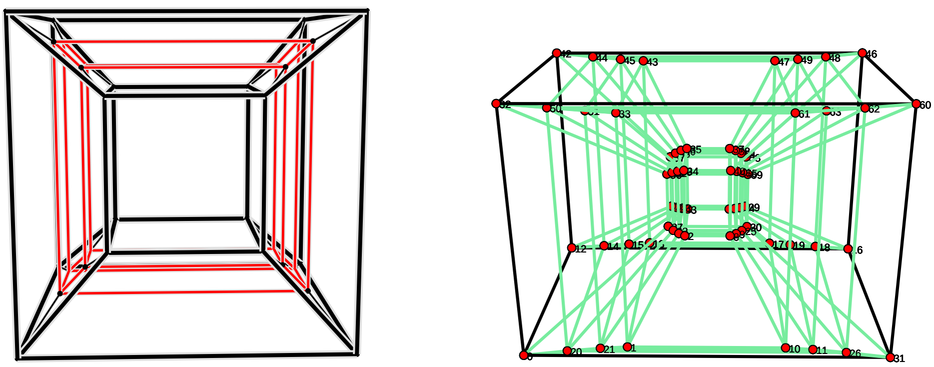
*Proof.* First, we determine the vertex figure $\mathsf{NCP}_d(Q)_{/v}$. Due to the fact that $v$ survives the projection, $v$ has a defining hyperplane $H(c, \delta)$ whose normal is of the form $c = (\mathbb{0}, c')^T$. Now, we if choose $\delta' < \delta$ appropriately, then $H(c, \delta')$ is a hyperplane that strictly separates $v$ from the other vertices in $\hat{C}_n(Q, d)$. But then $H(c', \delta')$ has the same (separation) property in $\mathsf{NCP}_d(Q)$.

The vertex figure is simplicial, since all faces containing $v$ that survive the projection are cubes, and the facial structure of $\mathsf{NCP}_d(Q)_{/v}$ is determined by the incidences of surviving faces containing $v$. By Theorem 3.14, these faces are determined by $\mathsf{Lex\text{-}Pyr}_p(Q)$, which proves the claim. Moreover, the edge $\alpha' = (0, \alpha_1, \dots, \alpha_n)$ survives the projection as well and it corresponds to the apex of $\mathsf{Lex\text{-}Pyr}_p(Q)$. If we take an *edge figure*, i.e. an iterated vertex figure, then this corresponds to the link of $v_0$ in $\mathsf{Lex\text{-}Pyr}_p(Q)$ which corresponds to $\mathcal{C}(\partial Q)$ by Definition 2.21. □

Before we close this section, we hope to enlighten the reader by pictures of what we have been dealing with. Figures 3.2(a) and 3.2(b) show Schlegel diagrams of the neighborly cubical polytopes $\mathsf{NCP}_4(5)$ and $\mathsf{NCP}_4(6)$ respectively. Needless to say that 3-dimensional visualizations of 4-dimensional objects drawn on 2-dimensional paper are expressionally unsatisfactory. We invite the reader to visit

`http://www.math.tu-berlin.de/~sanyal/diploma`

for an interactive version of the presented figures.



(a) $\mathsf{NCP}_4(5)$: A cubical 4-polytope with the graph of the 5-cube.

(b) $\mathsf{NCP}_4(6)$: A cubical 4-polytope with $f$-vector $(64, 192, 192, 64)$, which amounts to the fact that the Schlegel diagram is rather crowded.

Figure 3.2: Figures (a) and (b) show Schlegel diagrams of the neighborly cubical polytopes $\mathsf{NCP}_4(5)$ and $\mathsf{NCP}_4(6)$ respectively.

## 3.3 Deformed Products of Polygons

Fatness, on which we digressed at the beginning of this chapter, seems to be a driving force in the study of 4-dimensional polytopes. Ernst Steinitz, in his seminal 3 pages article from (1906), pioneered the study of $f$-vector of 3-polytopes by giving linear inequalities that are satisfied by every $f$-vector. Since then, many efforts have gone into the task of describing the cone of $f$-vectors of 4-polytopes. Till the present day, it seems that the right question[2] to ask is whether *fatness* is bounded for 4-polytopes. Space limitations make us refrain from giving further details but the interested reader might find satisfaction in Eppstein, Kuperberg, and Ziegler (2003) and Ziegler (2002).

Ziegler (2004) constructs 4-polytopes with fatness arbitrarily close to 9 by proving the following theorem.

**Theorem 3.18 (Ziegler (2004), Theorem 1.1).** *Let $n \geq 4$ be even and $r \geq 2$ then there is a $2r$-polytope $P_n^{2r} \subset \mathbb{R}^{2r}$, combinatorially equivalent to a product of $r$ $n$-gons, such that the projection $\pi_4 : \mathbb{R}^{2r} \to \mathbb{R}^4$ to the last four coordinates strictly preserves the 1-skeleton as well as all the "polygon 2-faces" of $P_n^{2r}$.*

The main purpose of this section is a generalized version of this theorem and an explicit combinatorial description of the projected polytopes.

In order to give an appealing account on the combinatorics of projected deformed products of polygons, we shall shortly discuss the description of polygons and their products.

For an even polygon $P_n$ with $n \geq 4$ ($n$ even) vertices pick one edge as a starting point and label the edges in clockwise order with $(*, i)$ if the $i$-th edge is odd and $(i, *)$ otherwise. The labels of the vertices then correspond to $(i, j)$ if the vertex is in the intersection of $(i, *)$ and $(*, j)$. The polygon itself is labeled $(*, *)$ as it is an improper face. See Figure 3.3 for a labeled 6-gon. So for an $n$-gon with $n$ even the face lattice, excluding the empty face, is given by

$$\begin{aligned} \mathcal{P}_n = {} & \{(i, i \pm 1) : i \in \mathbb{Z}_n \text{ even}\} && \text{(vertices)} \\ & \cup \{(i, *) : i \in \mathbb{Z}_n \text{ even}\} && \text{(even edges)} \\ & \cup \{(*, i) : i \in \mathbb{Z}_n \text{ odd}\} && \text{(odd edges)} \\ & \cup \{(*, *)\} && \text{(polygon)} \end{aligned}$$

[2] Apart from the question to which the answer is obviously 42.

Arithmetic operations in the description above are understood modulo $n$. In that notation, there are vertices that will become important later which is the reason why we give them special names. The vertices $(0,1)$ and $(0,n-1)$ will be called *leading*- and *trailing vertex* respectively. The remaining vertices will be henceforth referred to as *inner vertices*.

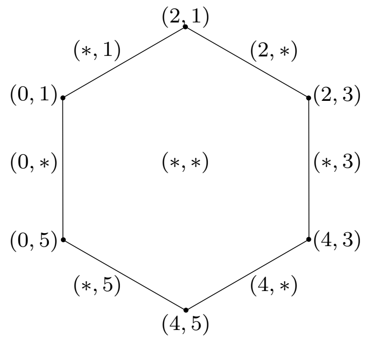



Figure 3.3: Labeling of a 6-gon

There is no doubt that this is a rather uncommon description of a polygon. The advantages, however, are that this description has the same "nice" properties that made the cube so easy to describe: given a face $(i,j)$ then the number of stars tells us its dimension. To fully appreciate the notation, let us form products of polygons and adapt our formalism to it. Let $n \geq 4$ and $r \geq 2$. Then the faces of $(P_n)^r$, the $r$-fold product of $n$-gons, are given by vectors

$$(\alpha_1, \alpha_2\ ;\ \alpha_3, \alpha_4\ ;\ \ldots\ ;\ \alpha_{2r-1}, \alpha_{2r}) \in (\mathcal{P}_n)^r.$$

The semicolons separate the factors from each other. As discussed in Section 1.3, if $F_1, \ldots, F_r$ are non-empty faces of each factor then $F := F_1 \times F_2 \times \cdots \times F_r$ is a face of the product $(P_n)^r$ and the dimension of $F$ is $\mathsf{dim}(F) = \sum_{i=1}^{r} \mathsf{dim}(F_i)$. Analogous to the cube, if $\alpha \in (\mathcal{P}_n)^r$ represents $F$, then the dimension of $F$ can be read off $\alpha$ by counting the stars.

The observant reader might have noticed that for the case of $P_n$ being a quadrilateral ($n = 4$) Theorem 3.18 was already proved in the last section. Indeed, the neighborly cubical polytope $\mathsf{NCP}_d(2r)$ is an $r$-fold product of quads and thus a deformed product of polygons. In the case of $d$ being even, the inequality system looks as follows:

$$
\left(\begin{array}{cccccc|cccc||l}
\pm\varepsilon & & & & & & & & & & b_1 \\
1 & \pm\varepsilon & & & & & & & & & b_2 \\
 & 1 & \pm\varepsilon & & & & & & & & b_3 \\
 & & 1 & \pm\varepsilon & & & & & & & b_4 \\
 & & & & \ddots & & & & & & \vdots \\
 & & & & \pm\varepsilon & & & & & & b_{2r-d-1} \\
 & & & & 1 & \pm\varepsilon & & & & & b_{2r-d} \\
 & & & & & 1 & \pm\varepsilon & & & & b_{2r-d+1} \\
\multicolumn{6}{c|}{\text{———} \; g_1 \; \text{———}} & & \pm\varepsilon & & & b_{2r-d+2} \\
\multicolumn{6}{c|}{\vdots} & & & \ddots & & \vdots \\
\multicolumn{6}{c|}{\text{———} \; g_{d-2} \; \text{———}} & & & \pm\varepsilon & & b_{2r-1} \\
\multicolumn{6}{c|}{\text{———} \; g_{d-1} \; \text{———}} & & & & \pm\varepsilon & b_{2r}
\end{array}\right)
\tag{3.2}
$$

A close examination reveals that the first few quads have a rather particular shape (see Figure 3.4). One way to interpret the shape is that the normals $(\pm\varepsilon, 0)$ arose as perturbations of the zero vector and $(1, \pm\varepsilon)$ originated from the vector $(1, 0)$.

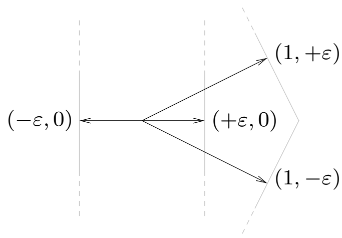



Figure 3.4: Scaled quad of an $\mathsf{NCP}_d(2r)$

The polygons that we will use for our construction of the Deformed Products of Polygons arise as perturbations of the above mentioned quads. We encourage the reader to retrace the construction at Figure 3.5. In addition to the normals of the quad $a_0, a_1, a_{n/2}$, and $a_{n-1}$ choose points on the line $L$ equally distributed above and below $a_{n/2}$ and choose scaling factors $b_1, b_2, \ldots, b_{n-1}$ such that the rescaled vectors $\frac{1}{b_i}a_i$ lie on the parabola $Q$ for $i = 1, \ldots, n-1$. Setting $b_0 = 1$, the points $\{\frac{1}{b_i}a_i : i = 0, \ldots, n-1\}$ are in convex position around the origin and thus the points $x \in \mathbb{R}^2$ satisfying

$$a_i^T x \leq b_i \text{ for } i = 0, \ldots, n-1$$

determine a convex $n$-gon in the plane.

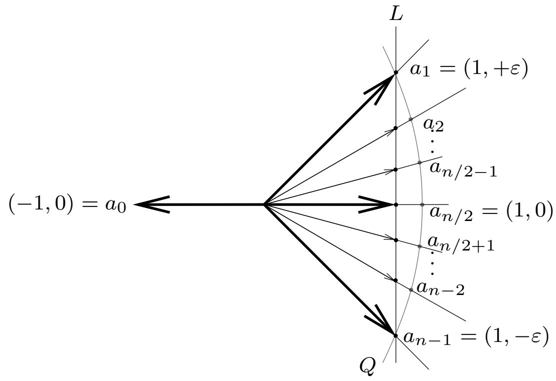



Figure 3.5: Bold vectors denote the quad used in $\mathsf{NCP}_d(2n)$. The remaining ones are those added to obtain a $n$-gon.

For the finishing touch, we scale every even-indexed inequality by $\varepsilon$ (see Figure 3.6) and arrange the scaled normals and right hand sides into a matrix $A \in \mathbb{R}^{n\times 2}$ and vector $b \in \mathbb{R}^n$, respectively.

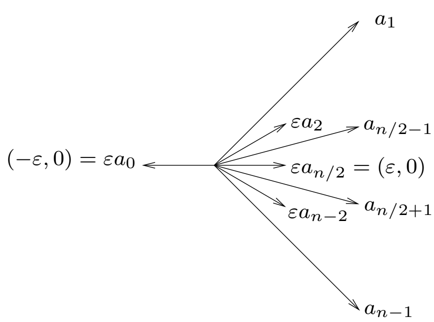



Figure 3.6: Rescaled normals: even indexed normals are scaled by $\varepsilon > 0$.

If we substitute the polygons for the quads in the inequality system (3.2) we obtain, by means of adaptations of the right hand sides $\mathsf{b_i}$, a deformed product of polygons which we denote by $\hat{P}^r_n(Q, d)$ shown in (3.3).

The description in (3.3) heavily relies on the reader's intuition. What might baffle the reader most is that we nonchalantly used both matrices and row vectors in the inequality system. This usage was meant to suggest the following: As can be seen in (3.2), if we arrange the normals of each quad in clockwise order, the odd and even indexed normals are subject to different perturbations. In the transition from quads to *even* polygons ($n > 4$) these distortions can be naturally extended, i.e. newly added normals get the perturbations according to their index parity. We hope the reader will indulge this pragmatic treatment.

$$\left(\begin{array}{cccc|ccc||c}
\boxed{A} & & & & & & & \mathsf{b}_1 \\
{}^{1} & \boxed{A} & & & & & & \mathsf{b}_2 \\
 & & \ddots & & & & & \vdots \\
 & & & \boxed{A} & & & & \mathsf{b}_2 \\
\multicolumn{4}{c|}{\overset{1}{\text{———}\; g_1 \;\text{———}}} & \boxed{A} & & & \mathsf{b}_2 \\
\multicolumn{4}{c|}{\vdots} & & \ddots & & \vdots \\
\multicolumn{4}{c|}{\begin{array}{c}\text{———}\; g_{d-2} \;\text{———} \\ \text{———}\; g_{d-1} \;\text{———}\end{array}} & & & \boxed{A} & \mathsf{b}_r
\end{array}\right) \tag{3.3}$$

The reason for $\hat{P}_n^r(Q,d)$ being a combinatorial product of polygons is that, again, perturbations of the facet normals can be countered by scaling the right hand side appropriately.

Having reached that point, we only have to zero in on that we proved the first half of the following generalization of a result of Ziegler (2004).

**Theorem 3.19 (Deformed Products of Polygons).** *Let $r \in \mathbb{N}$ and let $n \geq 4$ and $d = 2\ell \leq 2r$ be even. Then there exist $2r$-dimensional polytopes $\hat{P}_n^r(Q,d) \subset \mathbb{R}^{2r}$ combinatorially equivalent to $r$-fold products of $n$-gons whose projection $\mathsf{DPP}_d(Q,n) := \pi_d(\hat{P}_n^r(Q,d))$ to the last $d$ coordinates retains the $(\ell-1)$-skeleton.*

Note that Ziegler's original result asserts the survival of all "polygon 2-faces". These 2-faces are *special* in the sense that every edge is contained in an $n$-gon and thus retaining the $n$-gons implies the preservation of the 1-skeleton. Therefore the above result can be sharpened that *special* $\ell$-faces, that are faces that cover the $(\ell-1)$-skeleton, survive the projection.

We constructed the deformed products of polygons as an alteration of the neighborly cubical polytopes. From that it seems plausible that the projection of $\mathsf{DPP}_d(Q,n)$ exhibits the same structural properties as for $\mathsf{NCP}_d(Q)$.

Let $v$ be a vertex of $\hat{P}_n^r(Q,d)$ and let

$$\bar{G}(v) := \begin{pmatrix} a_1 & a_1' & & & & & & \\ 1 & a_2 & & & & & & \\ & 1 & a_3 & a_3' & & & & \\ & & 1 & a_4 & & & & \\ & & & \ddots & \ddots & & & \\ & & & & 1 & a_{2r-d-1} & a_{2r-d-1}' \\ & & & & & 1 & a_{2r-d} \\ & & & & & & 1 \\ \hline & & & & g_1 & & \\ & & & & \vdots & & \\ & & & & g_{d-1} & & \end{pmatrix} \in \mathbb{R}^{2r\times(2r-d)} \tag{3.4}$$

the matrix whose rows are the $2r-d$ components of normals to facets that contain $v$. We arranged the rows in a way as to match the combinatorial description of $v$. That means that every row corresponding to an even edge $(a_i, a_i')$ from a factor precedes the odd edge $(1, a_{i+1})$ from the same factor. By construction the entries $a_i$ and $a_i'$ satisfy

$$a_i \leq \varepsilon \text{ and}$$
$$a_i' < \varepsilon^2$$

for all $i \in [2r-d]$ and thus meet the requirements of Corollary 2.23, i.e. we can neglect the entries $a_i'$ in the further discussion.

The matrix $\bar{G}(v)$ therefore corresponds to a perturbed Gale transform and the facial structure of the polytope associated to $\bar{G}(v)$ matches that of $\mathsf{Lex\text{-}Pyr}_p(Q)$ with

$$p := \mathsf{min}(\{i \in [2r-d] : a_i > 0\} \cup \{2r-d\}).$$

Take a moment time and see that this already proves the theorem. The Gale transform $\bar{G}(v)$ corresponds to a perturbed pyramid over $Q$ and is thus a $\lfloor \frac{d-2}{2} \rfloor$-neighborly $(d-1)$-polytope. In consequence, every set of at most $\lfloor \frac{d}{2} \rfloor - 1$ rows might be deleted from $\bar{G}(v)$ such that the remaining rows are still positively spanning. By Corollary 3.4, this guarantees the *survival* of the corresponding face.

The last thing we aim at is to present to the reader the combinatorics of the projection in case $Q$ is the cyclic polytope $C_{d-2}(2r-1)$ with vertices in standard order. We denote the projection by $\mathsf{DPP}_d(n,r)$ in analogy to the previous section.

For the neighborly cubical polytopes, signs on the secondary diagonal $(a_1, \ldots, a_{2r-d})$ were in one-to-one correspondence with the combinatorial description of the vertex under consideration. For our combinatorial description of the polygon, this is unfortunately not the case.

We can recover the situation by introducing the following sign function.

$$\mathsf{sign} : \mathbb{Z}_n \cup \{*\} \to \{-, 0, +\}, i \mapsto \begin{cases} 0\,, \text{ if } i = * \\ -, \text{ if } i = 0 \\ +, \text{ if } i > 0 \text{ even} \\ +, \text{ if } i < \frac{n}{2} \text{ odd} \\ -, \text{ if } i > \frac{n}{2} \text{ odd.} \end{cases}$$

Admittedly, this is not the most elegant way but the reader might come to terms with it, after the following justification. By looking again at $\bar{G}(v)$ as depicted in (3.4), the reader can convince himself that every odd entry $a_i$ corresponds to the first component of an even edge of $P_n$ and every even entry corresponds to the second component of an odd edge. Now, by reinspecting Figure 3.5, the reader will find that $\mathsf{sign}$ corresponds to the signs of the respective components.

By extending $\mathsf{sign}$ componentwise to $(\mathcal{P}_n)^r$, this enables us to relate the combinatorics of $\mathsf{DPP}_d(n)$ to that of a neighborly cubical polytope.

**Theorem 3.20.** *Let $\alpha \in (\mathcal{P}_n)^r$ name a $(d-1)$-face $F$ of $\hat{P}_n^r(Q, d)$ with $Q = C_{d-2}(2r - \ 1\ )$. Then $F$ strictly survives the projection if and only if $\mathsf{sign}(\alpha)$ satisfies the Cubical Gale's Evenness Condition.*

Stating the conditions on $\alpha$ without reference to the Cubical Gale's Evenness Condition, then this results in the following four cases for $\alpha$. Let $\alpha \in (\mathcal{P}_n)^r$ name a $d-1$ dimensional face of $\hat{P}_n^r(Q, d)$ and let $p \geq 0$ be the least number with $\alpha_{p+1} = *$. Then $\alpha' = (\alpha_{p+2}, \ldots, \alpha_{2r})$ has to satisfy that for every $p+2 \leq i < j \leq 2r$ with $\alpha_i, \alpha_j \neq *$ the number of '$*$' entries $\#\{i < k < j : \alpha_k = *\}$ is even and

- $p = 0$ or $p = 2r - d + 1$
- $1 \leq p \leq 2r - d$ odd and $\alpha$ is of the form

$$\alpha = (0, n-1\,;\, 0, n-1\,;\, \ldots\,;\, \alpha_p, *\,; \alpha')$$

  - if $\alpha_p = 0$ then $\alpha'$ begins with an even number of $*$ entries, or
  - if $\alpha_p > 0$ even then $\alpha'$ begins with an odd number of $*$ entries.

- $1 \leq p \leq 2r - d$ even and $\alpha$ is of the form

$$\alpha = (0, n-1\,;\, 0, n-1\,;\, \ldots\,;\, 0, \alpha_p; *\,, \alpha')$$

- if $\alpha_p = n-1$ then $\alpha'$ begins with an even number of $*$ entries, or
- if $\alpha_p = 1$ even then $\alpha'$ begins with an odd number of $*$ entries.

No doubt, these conditions are even worse than those for the neighborly cubical polytopes, but that is combinatorics for you.

An example is more than appropriate and thus we exemplify the combinatorics at $\mathsf{DPP}_4(6,3)$

| $p$ | $(\alpha_1, \alpha_2 \,;\, \alpha_3, \alpha_4 \,;\, \alpha_5, \alpha_6)$ | | | | | | $\mathsf{sign}(\alpha_1, \alpha_2 \,;\, \alpha_3, \alpha_4 \,;\, \alpha_5, \alpha_6)$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $*$ | $*$ | $*$ | o | e | o | 0 | 0 | 0 | $\pm$ | $\pm$ | $\pm$ |
| | $*$ | o | $*$ | $*$ | e | o | 0 | $\pm$ | 0 | 0 | $\pm$ | $\pm$ |
| | $*$ | o | e | $*$ | $*$ | o | 0 | $\pm$ | $\pm$ | 0 | 0 | $\pm$ |
| | $*$ | o | e | o | $*$ | $*$ | 0 | $\pm$ | $\pm$ | $\pm$ | 0 | 0 |
| | $*$ | $*$ | e | o | e | $*$ | 0 | 0 | $\pm$ | $\pm$ | $\pm$ | 0 |
| 1 | e | $*$ | $*$ | $*$ | e | o | $+$ | 0 | 0 | 0 | $\pm$ | $\pm$ |
| | e | $*$ | e | $*$ | $*$ | o | $+$ | 0 | $\pm$ | 0 | 0 | $\pm$ |
| | e | $*$ | e | o | $*$ | $*$ | $+$ | 0 | $\pm$ | $\pm$ | 0 | 0 |
| | 0 | $*$ | $*$ | o | e | $*$ | $-$ | 0 | 0 | $\pm$ | $\pm$ | 0 |
| 2 | 0 | 1 | $*$ | $*$ | $*$ | o | $-$ | $+$ | 0 | 0 | 0 | $\pm$ |
| | 0 | 1 | $*$ | o | $*$ | $*$ | $-$ | $+$ | 0 | $\pm$ | 0 | 0 |
| | 0 | 5 | $*$ | $*$ | e | $*$ | $-$ | $-$ | 0 | 0 | $\pm$ | 0 |
| 3 | 0 | 5 | e | $*$ | $*$ | $*$ | $-$ | $-$ | $+$ | 0 | 0 | 0 |
| | 0 | 5 | 0 | $*$ | $*$ | $*$ | $-$ | $-$ | $-$ | 0 | 0 | 0 |

with $\mathsf{e} \in \{2,4\}$ and $\mathsf{o} \in \{1,3,5\}$. The right column of the table shows the value of the sign function on the corresponding face $\alpha$. The entry $\pm$ indicates that the sign depends on the corresponding value of e or o, respectively.

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# Zusammenfassung

Der Fokus dieser Diplomarbeit liegt in der Konstruktion und kombinatorischen Beschreibung von hoch-dimensionalen Polytopen. Polytope sind wohlbekannte Objekte der diskreten Geometrie. Eine Einführung in die Theorie der Polytope sowie gebräuchliche Begriffe und Notationen werden im ersten Kapitel gegeben.

Das zweite Kapitel gibt eine Übersicht zu Gale Transformierten und polyedrische Unterteilungen. Gale Transformierte werden klassischer Weise benutzt um hoch-dimensionale Polytope mit wenigen Ecken mit nieder dimensionalen Vektor Konfigurationen (gewöhnlich in der Ebene) zu assoziieren. Diese Vektor Konfigurationen besitzen in einem präzisen Sinne die gleiche kombinatorische Struktur und sind in den meisten Fällen handhabbarer. In dieser Arbeitfinden Gale Transformierte jedoch ein *neues* Anwendungsgebiet. Polyedrische Unterteilungen sind ein technischer Apparat um geometrische Objekte mittels Familien von Polytopen, mit speziellen Schnitteigenschaften, zu beschreiben. Wir verwenden Zeit (bzw. Platz) auf diese Konzepte, da es sich herausstellt, dass die im letzten Kapitel konstruierten Polytope mit diesen Mitteln auf eine natürliche Weise kombinatorisch beschrieben werden können.

Die zentrale Konstruktion dieser Arbeit ist die der sogenannten *deformierten Produkte*. Dabei handelt es sich um kartesische Produkte von Polytopen, die auf eine kontrollierte Weise mit Hilfe von Gale Transformierten deformiert werden. Die eigentlichen Polytope entstehen durch Projektion dieser Produkte. Für Polytope ist es im allgemeinen schwierig eine Aussage über die Struktur einer Projektion zu treffen. In den betrachten Fällen weisen wir nach, dass die Kombinatorik durch lexikographische Triangulierungen, speziellen polyedrischen Unterteilungen, von zyklischen Polytopen beschrieben werden kann. Wir illustrieren die Konstruktion an Hand von $d$-dimensionalen Würfeln, die als Produkte von Steckensegementen entstehen, und Produkten von Polygonen. Dies führt zu den in Joswig and Ziegler (2000) und Ziegler (2004) konstruierten Polytopen, deren Struktur sich mit den erarbeiteten Mitteln erstmals explizit beschreiben und nachvollziehen lässt.