Exercise sheet 5 for
Algebraic curves and the Weil conjectures

Kay Rülling

Exercise 5.1. Let $k$ be a perfect field and $C/k$ be a smooth irreducible projective curve over $k$ with function field $k(C)$. Take $t \in k(C)^\times$ a transcendental element over $k$ such that $k(C)/k(t)$ is a finite separable field extension.

1) Show that $\Omega^1_{k(C)/k} = k(C) \cdot dt$.

2) Show that if $P \in (C/k)_0$ is a closed point and $t_P \in \mathcal{O}_{C/k,P}$ is a local parameter (i.e. it generates the maximal ideal), then $\Omega^1_{k(C)/k} = k(C) \cdot dt_P$. (Hint: First show that $\Omega^1_{C/k,P} = \mathcal{O}_{C,P} \cdot dt_P$. To this end observe that we know already that $\Omega^1_{C/k,P}$ is locally free, hence (by a lemma from the lecture) we know, that it suffices to see that $dt_P \otimes 1$ is a basis of $\Omega^1_{C/k,P} \otimes \mathcal{O}_{C,P} k(P)$ which follows from a certain exact sequence for $\Omega^1$.)

3) Let $P \in (C/k)_0$ be a closed point and denote by $v_P : k(C)^\times \to \mathbb{Z}$ the associated discrete normalized valuation. Define $v_P(dt)$ as follows: take $t_P$ a local parameter at $P$ and write $dt = f_Pdt_P$ with $f_P \in k(C)^\times$ (see (2)); then $v_P(dt) := v_P(f_P)$. Show that this definition is independent of the choice of the local parameter $t_P$.

4) Define the Weil divisor $K_{C/k} := \sum_{P \in (C/k)_0} v_P(dt)[P] \in Z^1(C/k)$. Show that it is well-defined, i.e. the sum is finite.

5) Show that there is a natural isomorphism

$$\omega_{C/k} \cong \mathcal{O}_{C/k}(K_{C/k}),$$

where $\omega_{C/k} = \Omega^1_{C/k}$ and for $U \subset C/k$ open $\Gamma(U, \mathcal{O}_{C/k}(K_{C/k})) = \{f \in k(C)^\times \mid \text{Div}(f)|_U \geq -K_{C/k}\}$.

The divisor $K_{C/k}$ defined above is called the canonical divisor of $C/k$ and it is well defined as an element in $\text{CH}^1(C/k)$.

Exercise 5.2. Let $f : C' \to C$ be a dominant $k$-morphism between smooth connected curves $/k$.

¹Questions or comments to kay.ruelling@fu-berlin.de or come to 1.103(RUD25) on Tue/Thu/Fri.
(1) Let $Q \in (C'/k)_0$ be a closed point with image $P = f(Q) \in (C/k)_0$. Define the natural number $e_Q = e(Q/P)$ as follows: Take a local parameter $t_P \in \mathcal{O}_{C,P}$, then set

$$e_Q := e(Q/P) := v_Q(t_P),$$

where $v_Q : k(C')^\times \to \mathbb{Z}$ is the normalized discrete valuation corresponding to $Q$ and we view $t_P$ as an element of $k(C')$ via the inclusion $k(C) \hookrightarrow k(C')$ defined by $f$. Show that $e_Q$ is well-defined, i.e., independent of the choice of the parameter $t_P$.

(2) Let $D = \sum n_i [P_i], P_i \in (C/k)_0$, be a Weil divisor on $C$. Set $f^*D := \sum n_i \sum_{Q \in (f^{-1}(P_i))/k} e(Q/P_i)[Q]$. This defines a group homomorphism $f^* : Z^1(C/k) \to Z^1(C'/k) \to \text{CH}^1(C'/k)$, where the second map is the quotient map. Show that $f^*$ induces a well-defined map (again denoted by $f^*$) $\text{CH}^1(C/k) \to \text{CH}^1(C'/k)$.

**Exercise 5.3.** Let $f : C' \to C$ be a dominant $k$-morphism between smooth projective and connected curves over a perfect field $k$. Assume the corresponding function field extension $k(C')/k(C)$ is separable.

1. For $Q \in (C'/k)_0$ define $r_Q \in \mathbb{N}_0$ as follows (notation as above): Take $t_P$ a local parameter at $P = f(Q)$ and $t_Q$ a local parameter at $Q$ we can view $dt_P$ as an element in $\omega_{C'/k,Q}$ and write $dt_P = g_P dt_Q$ (see 5.1 (2)). Then we define

$$r_Q := v_Q(g_P).$$

Show that this definition is independent of the choices of $t_P$ and $t_Q$.

2. Let $p = \text{char}(k) \geq 0$. Assume $p = 0$ or $e_Q = e(Q/P)$ (see 5.2 (1)) is prime to $p$. Show that $r_Q = e_Q - 1$.

3. Show that in $\text{CH}^1(C'/k)$

$$[K_{C'}] = f^*[K_C] + [R],$$

where $R = \sum_{Q \in (C'/k)_0} r_Q \cdot [Q]$ is the ramification divisor.

**Exercise 5.4.** Let $\mathbb{P}^2(\bar{k}) = U_0 \cup U_1 \cup U_2$ be the standard open cover over $k$, i.e., $U_i = \mathbb{P}^2(\bar{k}) \setminus Z(X_i)$, where $X_0, X_1, X_2$ are the coordinates on $\mathbb{P}^2$.

1. Let $C = Z(F)$, where $F \in k[X_0, X_1, X_2]$ is an irreducible homogenous polynomial of degree $n$. We can view $C \subset \mathbb{P}^2/k$ as a prime Weil divisor and hence have the associated sheaf $\mathcal{O}_{\mathbb{P}^2}(-[C]) =: \mathcal{O}(-C)$. Show that $\mathcal{O}(-C)|_{U_i} = \mathcal{O}_{U_i} \cdot \frac{1}{X_i^n}$. (Hint: From commutative algebra we know that if $A = k[U]$ is the coordinate ring of a smooth affine $k$-variety with fraction field
$K$, then $A = \cap_{ht(p)=1} A_p$. Use this to show that $A = \{ f \in K \mid \text{div}_U(f) \geq 0 \}$. Conclude.)

(2) Show that there is an exact sequence

$$0 \to \mathcal{O}(-C) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_C \to 0,$$

i.e. $\mathcal{O}(-C)$ is the ideal sheaf of the embedding $C \hookrightarrow \mathbb{P}^2/k$.

(3) Show that there is an isomorphism $\mathcal{O}(-C) \cong \mathcal{O}_{\mathbb{P}^2/k}(-n)$.

(4) Show that

$$\dim_k H^1(C, \mathcal{O}_C) = \frac{(n-2)(n-1)}{2}.$$

(Hint: Use the short exact sequence from [2], the associated long exact sequence in cohomology and Theorem 7.3 from the lecture.)