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Exercise 8 for Number theory III[1]

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Exercise 8.1. Let k be a field and V a finite dimensional k-vector space. Let φ be a k-linear endomorphism of V. We view V as a $k[x]$-module by setting $x \cdot v := \varphi(v)$, $v \in V$. Show that V as a $k[x]$-module is simple if an only if the characteristic polynomial of φ is irreducible.

Exercise 8.2. Let k be a field and denote by $M_n(k)$ the k-algebra of $n \times n$-matrices. Show that $M_n(k)$ has center equal to k and is simple as a k-algebra, i.e. it has no non-zero proper two-sided ideals.

Exercise 8.3. Let k be a field of characteristic $\neq 2$ and $a, b \in k^\times$. Show that the following statements are equivalent:

(1) $\exists (x, y) \in k^2$ with $ax^2 + by^2 = 1$.
(2) $\exists (x, y, z) \in k^3 \setminus \{(0, 0, 0)\}$ with $ax^2 + by^2 = z^2$.
(3) $\exists (x, y, z, w) \in k^4 \setminus \{(0, 0, 0, 0)\}$ with $z^2 - ax^2 - by^2 + abw^2 = 0$.
(4) $\exists \gamma \in k(\sqrt{a})^\times$ with $b = \mathrm{Nm}_{k(\sqrt{a})/k}(\gamma)$.

(*Hint:* For (3) $\Rightarrow$ (4) show that if $\sqrt{a} \notin k$, then $\mathrm{Nm}(z + \sqrt{a}x) = b\mathrm{Nm}(y + \sqrt{a}w)$ and conclude.)

Exercise 8.4. Let k be a field of characteristic $\neq 2$ and $a, b \in k^\times$.

(1) Show that there is a unique (non-commutative) k-algebra $A(a, b; k)$ with generators α, β satisfying
$$\alpha^2 = a, \quad \beta^2 = b, \quad \alpha\beta = -\beta\alpha,$$
whose underlying vector space is the 4-dimensional k-vector space with basis $1, \alpha, \beta, \alpha\beta$ and whose center is equal to k.
(2) Show that if there are no elements $x, y \in k$ with $ax^2 + by^2 = 1$, then $A(a, b; k)$ is a divison algebra (or skew field), i.e. any non-zero element has a multiplicative inverse. (*Hint:* Use Exercise 8.3 (1) $\Leftrightarrow$ (3).)

[1]This exercise sheet will be discussed on December 12. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

(3) Show that if there exist $x, y \in k$ with $ax^2 + by^2 = 1$, then $A(a,b;k)$ is isomorphic as a k-algebra to the ring of 2×2-matrices with coefficients in k, i.e. $A(a,b;k) \cong M_2(k)$. In particular $A(a,b;k)$ is not a division algebra.

(*Hint:* If $\sqrt{a} \in k$, show that

$$\alpha \mapsto \begin{pmatrix} \sqrt{a} & 0 \\ 0 & -\sqrt{a} \end{pmatrix}, \quad \beta \mapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

defines an isomorphism $A(a,b;k) \cong M_2(k)$. If $[k(\sqrt{a}) : k] = 2$ denote by V the k-vector space $k(\sqrt{a})$, by $\mu_x : V \to V$ the multiplication by x map, i.e. $\mu_x(v) = xv$, let $\sigma \in G(k(\sqrt{a})/k)$ be the non-trivial element and take $\gamma \in k(\sqrt{a})$ with $\mathrm{Nm}(\gamma) = b$ (see Excercise 8.3). Then show that there is a unique isomorphism of k-algebras $A(a,b;k) \xrightarrow{\simeq} \mathrm{End}_{k-\mathrm{v.s.}}(V) \cong M_2(k)$ satisfying

$$1 \mapsto \mathrm{id}_V, \quad \alpha \mapsto \mu_\alpha, \quad \beta \mapsto \mu_\gamma \circ \sigma.)$$

Remark 1. A k-algebra of the form $A(a,b;k)$ as above is called a *quaternion algebra.* The divison algebra (!) $\mathbb{H} := A(-1,-1;\mathbb{R})$ is called the quaternion field of Hamilton.