Exercise 6 for Number theory III

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Exercise 6.1. Let \( K \) be a finite extension of \( \mathbb{Q}_p \). Recall that we equip \( K^\times \) with the topology for which \( U^{(n)}_K = 1 + m^n_K, \ n \geq 1 \), is a fundamental system of open neighborhoods of 1.

(1) Show that \( (K^\times)^n = \{a^n| a \in K^\times\} \) is an open subgroup of finite index in \( K^\times \), for all \( n \geq 1 \). (Hint: First show that for all \( n \geq 1 \) there are \( i, j \geq 1 \) with \( m^j_K = nm^i_K \) and such that the exponential function from Exercise 5.2 is defined on \( m^i_K \). Use the properties of \( \exp \) to conclude that \( (K^\times)^n \) is open. Use Exercise 4.2 to show that the index is finite.)

(2) Show that any subgroup of finite index of \( K^\times \) is open.

Exercise 6.2. Let \( K \) be a local field which is not \( \mathbb{R} \) or \( \mathbb{C} \). Use the Main Theorem of Local Class Field Theory to show that the finite unramified extensions of \( K \) are in one-to one correspondence with subgroups of finite index in \( K^\times/\mathcal{O}_K^\times \cong \mathbb{Z} \).

Exercise 6.3. Let \( p \neq 2 \) be a prime number. Show that there are exactly \( p + 1 \) abelian extensions of degree \( p \) over \( \mathbb{Q}_p \) only one of which is unramified. To this end proceed as follows:

(1) Use the Main Theorem of Local Class Field Theory and Exercise 6.1 to show that the number of abelian extensions of degree \( p \) over \( \mathbb{Q}_p \) is equal to the number of subgroups of \( \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)_p \).

(2) Show that \( \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)_p \cong \mathbb{Z}/p \times \mathbb{Z}/p \).

(3) Count the subgroups of index \( p \) in \( \mathbb{Z}/p \times \mathbb{Z}/p \).

(4) Use Exercise 6.2 to show that there is only one unramified abelian extension of degree \( p \) over \( \mathbb{Q}_p \).

Exercise 6.4. Let \( K \) be a local field, which is not \( \mathbb{R} \) or \( \mathbb{C} \). Let \( k = \mathbb{F}_q \) be its residue field. Recall from the lecture that we have the exact sequence

\[
0 \rightarrow I(K^{ab}/K) \rightarrow G(K^{ab}/K) \rightarrow G(k) \rightarrow 0,
\]

1This exercise sheet will be discussed on November 28. If you have questions or remarks please contact kay.ruelling@fu-berlin.de or kindler@math.fu-berlin.de or l.zhang@fu-berlin.de.
where $I(K^{ab}/K)$ denotes the inertia group and $G(k) \cong \hat{\mathbb{Z}}$ the absolute Galois group of $k$. We view $\mathbb{Z}$ as a subgroup of $G(k)$ by mapping $1 \in \mathbb{Z}$ to the $q$-power Frobenius. Denote by $d$ the composition $d : G(K^{ab}/K) \to G(K^{ur}/K) \cong G(k)$. We define the abelian Weil group of $K$ to be

$$W^{ab}(K) := d^{-1}(\mathbb{Z}).$$

It clearly contains $I(K^{ab}/K)$ and we equip it with the topology for which the open subsets of $I(K^{ab}/K)$ form a fundamental system of open neighborhoods of 1.

Show that the local Artin map $\rho_K : K^\times \to G(K^{ab}/K)$ induces an isomorphism of topological groups

$$\rho_K : K^\times \xrightarrow{\cong} W^{ab}(K)$$

under which $O_K^\times$ is mapped isomorphically to $I(K^{ab}/K)$.

(*Hint: Deduce this from the Main Theorem of Local Class Field Theory using that $O_K^\times$ and $I(K^{ab}/K)$ are both complete.*)