Exercise 4 for Number theory III

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Exercise 4.1. Let $K$ be a complete discrete valuation field with normalized discrete valuation $v : K^\times \to \mathbb{Z}$ and $L/K$ a finite separable field extension. We know from the lecture that $L$ is also a complete discrete valuation field. Show that its normalized discrete valuation is given by
\[ v_L : L^\times \to \mathbb{Z}, \quad a \mapsto \frac{1}{f} \cdot v(Nm_{L/K}(a)), \]
where $Nm_{L/K} : L^\times \to K^\times$ is the norm map and $f = f(L/K)$ is the inertia degree.

(Hint: To show $v_L(L^\times) \subset \mathbb{Z}$, let $E$ be the maximal unramified subextension of $L/K$ and use $Nm_{L/K} = Nm_{E/K} \circ Nm_{L/E}$.)

Exercise 4.2. Let $K$ be a local field (not $\mathbb{R}$, $\mathbb{C}$) and let $q = p^n$ be the cardinality of its residue field. Set $\mu_{q-1}(K) := \{a \in K^\times \mid a^{q-1} = 1\}$.

(1) Show that the natural surjection $\mathcal{O}_K \to \mathbb{F}_q$ induces a bijection of groups $\mu_{q-1}(K) \to \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$. (Hint: Hensel’s Lemma.)

(2) Let $\pi \in \mathcal{O}_K$ be a local parameter. Show that the group $K^\times$ admits a canonical decomposition
\[ K^\times \cong \pi^\mathbb{Z} \times \mu_{q-1}(K) \times U_K^{(1)}, \]
where $U_K^{(1)} = 1 + \pi \mathcal{O}_K$.

(3) Show that if $a \in K^\times$ has finite order $n$ (i.e. the group $\{1, a, a^2, \ldots \}$ has cardinality $n$), then $n|q-1$.

Exercise 4.3. Recall that we proved the following in Number Theory 2: Let $\zeta \in \overline{\mathbb{Q}}$ be a $p^r$-th primitive root of unity. Then
\[ (1) \ [\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(p^r) := (p-1)p^{r-1}. \]
\[ (2) \ \mathcal{O}_{\mathbb{Q}(\zeta)} = \mathbb{Z}[\zeta]. \]
\[ (3) \ p\mathbb{Z}[\zeta] = (1-\zeta)^{\varphi(p^r)}. \]

Show:

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1This exercise sheet will be discussed on November 14. If you have questions or remarks please contact kay.ruelling@fu-berlin.de or kindler@math.fu-berlin.de or l.zhang@fu-berlin.de.
(1) The same conclusion holds when we replace $\mathbb{Q}$ by $\mathbb{Q}_p$ and $\mathbb{Z}$ by $\mathbb{Z}_p$.

(2) There is a canonical decomposition

$$\mathbb{Q}_p(\zeta) \cong (1 - \zeta)\mathbb{Z} \times \mathbb{Z}/(p - 1)\mathbb{Z} \times U^{(1)}_{\mathbb{Q}_p(\zeta)}.$$

**Exercise 4.4.** Let $K$ be a finite extension of $\mathbb{Q}_p$. We know that it is a complete discrete valuation field. Let $A, m, v_K$ be its ring of integers, its maximal ideal and its normalized discrete valuation.

(1) Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements in $K$ and assume $v_K(a_n) \to \infty$, for $n \to \infty$. Show that the sum $\sum_{n=1}^{\infty} a_n$ converges, i.e. there exists a unique element $s \in K$ such that $s \equiv \sum_{n=1}^{\infty} a_n \mod m^N$ for all $N \geq 1$. Notice that by assumption the sum is finite modulo $m^N$. (In terms of the non-archimedean absolute value $|\cdot|_{v_K}$ defined in Exercise 1.1 one can rephrase this by saying: If $(a_n)_n$ is a null sequence in $K$ with respect to $|\cdot|_{v_K}$ then the sequence $(\sum_{n=1}^{N} a_n)_N$ converges in $K$.)

(2) Show that for $x \in m$ the sum $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ converges. We set

$$\log(1 + x) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad x \in m.$$

(3) Show that we obtain a continuous group homomorphism

$$\log : U^{(1)}_K \to K, \quad 1 + x \mapsto \log(1 + x).$$

Here we equip $U^{(1)}_K$ with the topology which is uniquely determined by the property that $U^{(1)}_K$ is a topological group and the sets $U^{(n)}_K := 1 + m^n, \ n \geq 1$, form a fundamental system of open neighborhoods of 1 and similar $K$ is the topological group with $m^n, \ n \geq 1$, as a fundamental system of open neighborhoods.

(4) Show that there is a continuous homomorphism

$$\log : K^\times \to K,$$

which is uniquely determined by the properties that $\log_U^{(1)}$ is the map from (3) and $\log(p) = 0$. (Hint: Use Exercise 4.2.)