27. Oktober 2014

Exercise 3 for Number theory III[1]

Kay Rülling

Exercise 3.1. Let K be any field and $K \hookrightarrow \bar{K}$ an embedding of K into an algebraic closure. Let L/K be an algebraic field extension.

(1) Show that any K-linear homomorphism of fields $L \to \bar{K}$ can be extended to an K-linear automorphism of $\bar{K}$.
(*Hint*: Use Zorn's Lemma.)
(2) Show that the following conditions are equivalent (in which case we say that L/K is Galois):
 (a) L/K is separable and $L^{\mathrm{Aut}_K(L)} = K$.
 (b) L/K is separable and any polynomial $f \in K[x]$ that has one root in L has all its roots in L.
(3) Let $E \subset L$ be an intermediate extension of L/K and assume that L/K and E/K are Galois. Show that there is a natural surjection $\mathrm{Gal}(L/K) \to \mathrm{Gal}(E/K)$ sending an element τ to its restriction $\tau_{|E}$.

Exercise 3.2. Let A be a Dedekind domain with fraction field K and L/K a finite Galois extension. Denote by B the integral closure of A in L. Let $\mathfrak{p} \subset A$ be a non-zero prime ideal of A.

Show that the Galois group $G = \mathrm{Gal}(L/K)$ acts transitively on the set of non-zero prime ideals $\mathfrak{q} \subset B$ which lie over $\mathfrak{p}$ (i.e. $\mathfrak{q} \cap A = \mathfrak{p}$).

To this end assume there exist $\mathfrak{q}, \mathfrak{q}' \subset B$ over $\mathfrak{p}$ such that $\sigma(\mathfrak{q}) \neq \mathfrak{q}'$, for all $\sigma \in G$ and show that this leads to a contradiction as follows:

(1) Show that there exists an element $x \in B$ with $x \equiv 0 \bmod \mathfrak{q}'$ and $x \equiv 1 \bmod \sigma(\mathfrak{q})$, for all $\sigma \in G$.
(2) Let $\mathrm{Nm}_{L/K} : L \to K$ be the norm map. Recall that $\mathrm{Nm}_{L/K}(y) = \prod_{\sigma \in G} \sigma(y)$, for $y \in L$. Show that it induces a multiplicative map $\mathrm{Nm}_{B/A} : B \to A$, which maps any non-zero prime ideal $\mathfrak{r} \subset B$ over $\mathfrak{p}$ into $\mathfrak{p}$.
(3) Deduce the contradiction by showing that one condition on x in (1) implies $\mathrm{Nm}_{B/A}(x) \in \mathfrak{p}$, while the other implies $\mathrm{Nm}_{B/A}(x) \notin \mathfrak{p}$.

[1] This exercise sheet will be discussed on November 7. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de`

Exercise 3.3. Let $\mathfrak{p} \subset A$, L/K, G and B be as in Exercise 3.2 above. Let $\mathfrak{q}$, $\mathfrak{q}' \subset B$ be two prime ideals over $\mathfrak{p}$. Denote by $D_\mathfrak{q}$ and $D_{\mathfrak{q}'} \subset G$ the respective decomposition groups.

(1) Show that there exists a $\sigma \in G$ such that $D_\mathfrak{q} = \sigma \circ D_{\mathfrak{q}'} \circ \sigma^{-1}$.
(2) Show that the index of $D_\mathfrak{q}$ in G is equal to the number of prime ideals in B lying over $\mathfrak{p}$.

Exercise 3.4. Let $\mathfrak{p} \subset A$, K be as in Exercise 3.2. Let $K_\mathfrak{p}$ be the completion of K along $\mathfrak{p}$. Fix algebraic closures $K \hookrightarrow \bar{K}$ and $K_\mathfrak{p} \hookrightarrow \bar{K}_\mathfrak{p}$ and denote by $K^{\mathrm{sep}} \subset \bar{K}$ and $K_\mathfrak{p}^{\mathrm{sep}} \subset \bar{K}_\mathfrak{p}$ the separable closure of K and $K_\mathfrak{p}$ and by $G(K)$ and $G(K_\mathfrak{p})$ the corresponding absolute Galois groups, respectively. Choose of an embedding $\iota : \bar{K} \hookrightarrow \bar{K}_\mathfrak{p}$.

(1) Show that ι induces an injective group homomorphism $G(K_\mathfrak{p}) \hookrightarrow G(K)$.
(2) Let L/K be a finite Galois extension inside $\bar{K}$ and B the integral closure of A in L. Show that there exists a unique prime $\mathfrak{q}_\iota \subset B$ lying over $\mathfrak{p}$ such that the composition $L \hookrightarrow \bar{K} \xrightarrow{\iota} \bar{K}_\mathfrak{p}$ factors like $L \hookrightarrow L_\mathfrak{q} \hookrightarrow \bar{K}_\mathfrak{p}$, where the first map is the natural one and the second comes from viewing $L_\mathfrak{q}$ as a subextension of $\bar{K}_\mathfrak{p}/K_\mathfrak{p}$.
(3) With $\mathfrak{q}_\iota$ as in (2) above show that there is a commutative diagram

$$\begin{array}{ccc} G(K_\mathfrak{p}) & \xrightarrow{(1)} & G(K) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(L_{\mathfrak{q}_\iota}/K_\mathfrak{p}) & \longrightarrow & \mathrm{Gal}(L/K), \end{array}$$

where the vertical maps are the natural quotient maps.
(4) Denote by $D_{\mathfrak{p},\iota}(K) \subset G(K)$ the image of the map from (1). Show that if we choose another embedding $\iota' : \bar{K} \hookrightarrow \bar{K}_\mathfrak{p}$, then there exists $\sigma \in G(K)$ such that $\sigma \circ D_{\mathfrak{p},\iota}(K) \circ \sigma^{-1} = D_{\mathfrak{p},\iota}(K)$.

Remark 1. The group $D_{\mathfrak{p},\iota}$ in the exercise above is called absolute decomposition group with respect to ι. By part (2) choosing ι corresponds to choose a compatible system of primes over $\mathfrak{p}$ in the finite Galois extension over K. By part (4) the image of $D_{\mathfrak{p},\iota}(K)$ in the abelian absolute Galois group $G^{\mathrm{ab}}(K)$ does not depend on the choice of ι. We denote this image by $D_\mathfrak{p}^{\mathrm{ab}}(K)$.