Exercise 2 for Number theory III

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Exercise 2.1. Let $I$ be a directed set (i.e. there is a partial ordering $\leq$ on $I$ such that for any two elements $i, j \in I$ there exists $n \in I$ with $n \geq i, j$). We say that a family $(G_i)_{i \in I}$ is a projective system of groups, if the $G_i$'s are groups and for any $j \geq i$ there exists a group homomorphism $\varphi_{j,i} : G_j \to G_i$ such that $\varphi_{i,i} = \text{id}_{G_i}$ and $\varphi_{j,i} \circ \varphi_{k,j} = \varphi_{k,i}$ for all $k \geq j \geq i$. For such a projective system define

$$
\lim_{\leftarrow \atop i \in I} G_i := \{ (g_i) \in \prod_{i \in I} G_i \mid \varphi_{j,i}(g_j) = g_i \text{ for all } j \geq i \}.
$$

Show that $\lim_{\leftarrow \atop i \in I} G_i$ has the following universal property: Let $H$ be a group and $h_i : H \to G_i$, $i \in I$, group homomorphisms such that $\varphi_{j,i} \circ h_j = h_i$, for all $j \geq i$. Then there exists a unique morphism $h : H \to \lim_{\leftarrow \atop i \in I} G_i$, which when composed with the natural projection maps $\lim_{\leftarrow \atop i \in I} G_i \to G_j$ equals $h_j$ for all $j \in I$.

Remark: $\lim_{\leftarrow \atop i \in I} G_i$ is called the projective limit of the projective system $(G_i)_{i \in I}$. It can be defined in a similar way if we replace groups by rings, modules, etc. (In general it exists by definition in a complete category.)

Exercise 2.2. Let $G$ be a topological group. Show:

1. Any open subgroup of $G$ is also closed.
2. Assume $G$ is quasi-compact (i.e. any open cover of $G$ has a finite refinement). Then any open subgroup $U$ of $G$ has finite index in $G$ (i.e. there are only finitely many left cosets of $U$ in $G$).
3. Assume $G$ is profinite. Then $G$ is compact (i.e. Hausdorff and quasi-compact) and totally disconnected (i.e. each point $g \in G$ is its own connected component.)

Remark: One can also show that if $G$ is a topological group that is compact and totally disconnected then it is profinite.

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*This exercise sheet will be discussed on October 31. If you have questions or remarks please contact kay.ruelling@fu-berlin.de or kindler@math.fu-berlin.de.*
Exercise 2.3. (1) Let $p$ be a prime number and $\mathbb{F}_q$ be a field with $q = p^r$ Elements. Fix an embedding $\mathbb{F}_q \hookrightarrow \mathbb{F}$ into an algebraic closure.

Show that there is an isomorphism of topological groups $\text{Gal}(\mathbb{F}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$.

(*Hint: You can use that for $n \geq 1$ the Galois group $\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is the cyclic group which is generated by the $r$-power Frobenius.*)

(2) Find a subgroup $H \subset \text{Gal}(\mathbb{F}/\mathbb{F}_q)$ such that $\text{Gal}(\mathbb{F}/\mathbb{F}_q)/H$ is not the Galois group of a Galois extension of $\mathbb{F}_q$. (Notice that by the Galois correspondence this $H$ cannot be a closed subgroup.)

(3) Let $\bar{\mathbb{Q}}$ be the subfield of $\mathbb{C}$ defined by adjoining all $n$-th roots of unity, $n \in \mathbb{N}$, to $\mathbb{Q}$.

Show that $\bar{\mathbb{Q}}/\mathbb{Q}$ is Galois and that there is an isomorphism of topological groups $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times$.

(*Hint: You can use that the Galois group of the $n$-th cyclotomic extension of $\mathbb{Q}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.*)

Exercise 2.4. Let $G$ be a profinite group. Denote by $G'$ the closure in $G$ of the subgroup generated by all the commutators $aba^{-1}b^{-1}$, with $a, b \in G$.

(1) Show that $G'$ is a normal subgroup of $G$. Hence we get a group $G^{ab} := G/G'$.

(2) Show that the group $G^{ab}$ is abelian and has the following universal property: Any continuous group homomorphism $\varphi : G \rightarrow H$ from $G$ into an *abelian* profinite group $H$ factors uniquely via a morphism $\varphi^{ab} : G^{ab} \rightarrow H$.

(3) Assume $G = \varprojlim_{i \in I} G_i$ (as topological groups), where $(G_i)_{i \in I}$ is a projective system of finite groups. Show $G^{ab} = \varprojlim_{i \in I} G_i^{ab}$.