Exercise 13 for Number theory III

Kay Rülling

Exercise 13.1. Let \( L \) be a field we define the second Milnor \( K \)-theory of \( L \) to be
\[
K_2^M(L) = L^\times \otimes_{\mathbb{Z}} L^\times / < a \otimes (1 - a) \mid a \in L \setminus \{0, 1\} >.
\]
The class of an element \( a \otimes b \) in \( K_2^M(L) \) is denoted by \( \{a, b\} \). Show that the following relations hold:
1. \( \{aa', b\} = \{a, b\} + \{a', b\}, \{a, bb'\} = \{a, b\} + \{a, b'\}, \{a, 1 - a\} = 0. \)
2. \( \{a, -a\} = 0, \{a, b\} = -\{b, a\}. \) (Hint: To show the first equality consider the identity \( -a = (1 - a) / (1 - a - 1) \); the second equality follows from the first.)

Exercise 13.2. Let \( K \) be a local field (\( \neq \mathbb{C}, \mathbb{R} \)). Let \( p \neq 2 \) be the characteristic of the residue field of \( K \) and \( n \in \mathbb{N} \) with \( (n, p) = 1 \).
1. Assume that \( K \) contains an \( n \)-th primitive root of unity \( \zeta \).
   (1) Show that for any \( a \in K^\times \) there exists a unique continuous homomorphism \( \chi_a : G(K_{\text{sep}}/K) \to \mathbb{Q}/\mathbb{Z} \) of order \( d \mid n \) which (see Exercise 11.2 (1)) corresponds to \( (K(\sqrt[n]{a}), \sigma) \), where \( K(\sqrt[n]{a}) \) denotes the splitting field of \( X^n - a \) and \( \sigma_a \in G(K(\sqrt[n]{a})/K) \) is the element, which on a root \( \alpha \) of \( X^n - a \) acts via \( \sigma_a(\alpha) = \zeta^n \alpha \).
   (2) Show that for \( a, a' \in K^\times \) we have \( \chi_{aa'} = \chi_a + \chi_{a'} \).

We define the map
\[
(\cdot, \cdot)_{n,K} : K^\times \times K^\times \to \frac{1}{n} \mathbb{Z}/\mathbb{Z}, \quad (a, b) \mapsto (a, b)_{n,K} := \text{inv}(A(\chi_a, b)),
\]
where \( \text{inv} : \text{Br}(K) \xrightarrow{\cong} \mathbb{Q}/\mathbb{Z} \) is the isomorphism from the lecture.
3. Show that \( (\cdot, \cdot)_{n,K} \) is \( \mathbb{Z} \)-bilinear, i.e.
\[
(aa', b)_{n,K} = (a, b)_{n,K} + (a', b)_{n,K}, \quad (a, bb')_{n,K} = (a, b)_{n,K} + (a, b')_{n,K}.
\]
4. Show that \( (a, 1 - a)_{n,K} = 0. \) (Hint: Let \( \alpha \in K(\sqrt[n]{a}) \) be a solution of \( X^n - a \) and show that \( 1 - a = 1 - \alpha^n = \prod_{i=0}^{n-1} (1 - \zeta^i \alpha) \) is contained in \( \text{Nm}_{K(\sqrt[n]{a})/K}(K(\sqrt[n]{a})^\times) \).

\[\text{This exercise sheet will be discussed on January 30. If you have questions or remarks please contact kay.ruelling@fu-berlin.de or kindler@math.fu-berlin.de or l.zhang@fu-berlin.de}\]
(5) Conclude that \((-,-)_{n,K}\) induces a homomorphism
\[ h_{n,K} : K_2^M(K)/n \cdot K_2^M(K) \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}, \quad \{a,b\} \mapsto (a,b)_{n,K}. \]

(6) Show that if \(K(\sqrt[n]{a})\) is unramified of degree \(d|n\) over \(K\), then \(h_{n,K}(\{a,b\}) = v_K(b)/d \text{ in } \frac{1}{n}\mathbb{Z}/\mathbb{Z}\).

(7) Conclude from (6) that \(h_{n,k}\) is surjective.

(8) Let \(\pi \in \mathcal{O}_K\) be a local parameter and \(q = p^s\) the cardinality of the residue field. Recall from Exercise 4.2, that we have \(K^\times = \mathbb{Z}^\times \times \mu_{q-1}(K) \times U_K^{(1)}\) and that by assumption \(n|(q-1)\). Let \(\xi \in \mu_{q-1}\) a primitive \((q-1)\)th root of unity.

Show that as a group \(K_2^M(K)/nK_2^M(K)\) is generated by the element \(\{\pi,\xi\}\) and \(\{\xi,-1\}\). (Hint: By Hensel’s Lemma any element in \(U_K^{(1)}\) is an \(n\)th power; use the relations from Exercise 13.1)

(9) Show that \(\mu_{q-1}(K) \cap \text{Nm}_K(\sqrt[n]{\pi}/K(\sqrt[n]{\pi})^\times) \subset (K^\times)^n\).

(10) Assume that \(2n|(q-1)\). Show that \(\{\xi,-1\} \equiv 0 \text{ mod } nK_2^M(K)\).

Conclude that \(h_{n,K} : K_2^M(K)/nK_2^M(K) \xrightarrow{\sim} \frac{1}{n}\mathbb{Z}/\mathbb{Z}\)

is an isomorphism.

**Remark 1.** Notice that in the situation above we have \(\frac{1}{n}\mathbb{Z}/\mathbb{Z} = \text{Br}(K)[n]\).

One can show that under the above assumptions \(\text{Br}(K)[n]\) is isomorphic to \(H^2(G(k_{\text{sep}}/k),\mu_n(k_{\text{sep}})^{\otimes 2})\) (by Hilbert 90). Thus we obtain an isomorphism

\[ h_{n,K} : K_2^M(K)/nK_2^M(K) \xrightarrow{\sim} H^2(G(k_{\text{sep}}/k),\mu_n(k_{\text{sep}})^{\otimes 2}). \]

It was proven by Suslin-Merkurjev in 1982 that there is such an isomorphism for all fields \(K\) and all natural numbers \(n\), which are invertible in \(K^\times\).

**Exercise 13.3.** Let \(L\) be a global field and \(p \subset \mathcal{O}_L\) a maximal prime. Denote by \(L_p\) the completion of \(L\) along \(p\) and by \(\pi \in p\mathcal{O}_{L_p}\) a local parameter. Let \(n\) be a natural number which is prime to the residue characteristic of \(L_p\). We assume that \(L\) contains an \(n\)-th primitive root of unity. We define the \(n\)-th power residue symbol to be the following map

\[ \mathcal{O}_L^\times \to \frac{1}{n}\mathbb{Z}/\mathbb{Z}, \quad u \mapsto \left(\frac{u}{p}\right) := h_{n,K}(\{\pi,u\}), \]

where we use the notation from Exercise 13.2.
(1) Show that
\[ \left( \frac{u}{p} \right) = 1 \iff u \text{ is an } n\text{th power modulo } p. \]

(2) Show that for \( L = \mathbb{Q} \), \( p = p \) an odd prime and \( n = 2 \), the symbol defined above is equal to the Legendre symbol from Exercise 7.2.