11. January 2015

Exercise 12 for Number theory III[1]

Kay Rülling

Exercise 12.1. Let G be a finite group (written multiplicatively) and A an abelian group (written additively) with G action

$$G \times A \to A, \quad (g, a) \mapsto g \cdot a.$$

Set $C^0(G, A) := A$ and $C^i(G, A) := \{\text{set maps } \varphi : \prod_{j=1}^{i} G \to A\}$, $i \geq 1$. The addition on A induces a structure of an abelian group on $C^i(G, A)$, all i, and we define group homomorphisms $\partial^i : C^i(G, A) \to C^{i+1}(G, A)$, $i = 0, 1, 2$, by

$$\partial^0(a)(g) := g \cdot a - a,$$

$$\partial^1(\alpha)(g, h) := g \cdot \alpha(h) - \alpha(gh) + \alpha(g),$$

$$\partial^2(\varphi)(f, g, h) := f \cdot \varphi(g, h) - \varphi(fg, h) + \varphi(f, gh) - \varphi(f, g),$$

for $a \in A$, $f, g, h \in G$, $\alpha \in C^1(G, A)$, $\varphi \in C^2(G, A)$.

Set $B^0(G, A) := 0$ and

$$B^i(G, A) := \mathrm{Im}(\partial^{i-1}), i = 1, 2, \quad Z^i(G, A) := \mathrm{Ker}(\partial^i), i \in [0, 2].$$

(1) Show $\partial^{i+1} \circ \partial^i = 0$, $i = 0, 1$. In particular $B^i(G, A) \subset Z^i(G, A)$ and we set

$$H^i(G, A) := Z^i(G, A)/B^i(G, A), \quad i = 0, 1, 2.$$

(2) Show that $H^0(G, A) = A^G$ and that if G acts trivial on A (i.e. $g \cdot a = a$ for all $g \in G$ and $a \in A$), then $H^1(G, A) = \mathrm{Hom}(G, A) = \mathrm{Hom}(G^{\mathrm{ab}}, A)$, where $\mathrm{Hom}(-, -)$ denotes group homomorphisms and G^{ab} is the maximal abelian quotient of G.

(3) Let $r : A \to B$ be a G-equivariant group homomorphism (i.e. $r(g \cdot a + a') = g \cdot r(a) + r(a')$, $g \in G$, $a, a' \in A$). Show that the composition $C^i(G, A) \to C^i(G, B)$, $\varphi \mapsto r \circ \varphi$ induces a group homomorphism

$$r^{(i)} : H^i(G, A) \to H^i(G, B), \quad i = 0, 1, 2.$$

[1]This exercise sheet will be discussed on January 23. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

(4) Let $0 \to A \xrightarrow{r} B \xrightarrow{s} C \to 0$ be a short exact sequence of abelian groups with G action and assume the maps r and s are G-equivariant. Show that there is a group homomorphism

$$\delta^i : H^i(G, C) \to H^{i+1}(G, A), \quad i = 0, 1$$

constructed as follows:

(a) For $\varphi \in Z^i(G, C)$ show there is an element $\tilde{\varphi} \in C^i(G, B)$ with $\varphi = s \circ \tilde{\varphi}$.

(b) Show that the element $\partial^i(\tilde{\varphi}) \in C^{i+1}(G, B)$ maps to zero under $C^{i+1}(G, B) \xrightarrow{s\circ} C^{i+1}(G, C)$.

(c) Conclude that there is a unique element $\psi \in C^{i+1}(G, A)$ with $r \circ \psi = \partial^i(\tilde{\varphi})$.

(d) Show that $\psi \in Z^{i+1}(G, A)$.

(e) Show that the class ψ in $H^{i+1}(G, A)$ is independent from the choice of $\tilde{\varphi}$ in (a).

(f) Show that this construction induces a well-defined map δ^i.

(5) Let $H \lhd G$ be a normal subgroup. Notice that we get a natural G/H-action on A^H and there is a natural map $C^i(G/H, A^H) \to C^i(G, A)$ induced by precomposing with the natural surjection $\prod^i G \to \prod^i G/H$ and composing with the inclusion $A^H \subset A$. Show that this induces a natural map (called the inflation map)

$$\mathrm{Inf} = \mathrm{Inf}_{G/H}^{G} : H^i(G/H, A^H) \to H^i(G, A), \quad i \in [0, 2].$$

(6) Show that if $H' \lhd H \lhd G$ is a chain of normal subgroups, then

$$\mathrm{Inf}_{G/H}^{G} = \mathrm{Inf}_{G/H'}^{G} \circ \mathrm{Inf}_{G/H}^{G/H'}.$$

(7) Let $H \lhd G$ be a normal subgroup. Assume we have a exact sequence as in (4) and that H acts trivially on A, B, C. Show that we have a commutative diagram for $i = 0, 1$

$$\begin{array}{ccc} H^i(G/H, C) & \xrightarrow{\delta^i} & H^{i+1}(G/H, A) \\ \downarrow{\scriptstyle \mathrm{Inf}} & & \downarrow{\scriptstyle \mathrm{Inf}} \\ H^i(G, C) & \xrightarrow{\delta^i} & H^{i+1}(G, A). \end{array}$$

Exercise 12.2. Let k be a field and L/k a finite Galois extension with Galois group $G = G(L/k)$. Recall from the lecture that we have an isomorphism $H^2(G, L^\times) \xrightarrow{\simeq} \mathrm{Br}(L/k)$, $[\varphi] \mapsto [A(\varphi)]$.

(1) Let $\varphi : G \times G \to L^\times$ be a normalized 2-cocycle. Let $\{e_\sigma\}_{\sigma \in G}$ be an L-basis of $A(\varphi)$ satisfying $e_\sigma \cdot \lambda = \sigma(\lambda) e_\sigma$, for $\lambda \in L$, and $e_\sigma \cdot e_\tau = \varphi(\sigma, \tau) e_{\sigma\tau}$. We form the left L-vector space $A(\varphi) \otimes_L$

$M_m(L)$, where the tensor product is formed using the *left* L-vector space structure of $A(\varphi)$.

(a) Show that there is a unique k-algebra structure on $A(\varphi)\otimes_L M_m(L)$ such that

$$(e_\sigma \otimes \alpha)\cdot(e_\tau \otimes \beta) = e_\sigma e_\tau \otimes \alpha\sigma(\beta).$$

(b) Show that there is an isomorphism of k-algebras $A(\varphi)\otimes_k M_m(k) \cong A(\varphi)\otimes_L M_m(L)$.

(2) Let L'/L be another finite Galois extension of degree m. Set $G' = G(L'/k)$ and denote by $G' \to G$, $\sigma \mapsto \bar{\sigma}$ the quotient map. Fix a basis of L'/L and denote by $\mu_\sigma \in M_m(L)$ the matrix obtained by the action of σ on this fixed basis. (Be aware that σ is not L-linear!) Show that there is an isomorphism of k-algebras

$$A(\mathrm{Inf}_{G(L/k)}^{G(L'/k)}(\varphi)) \xrightarrow{\simeq} A(\varphi)\otimes_L M_m(L), \quad e_\sigma \mapsto e_{\bar{\sigma}} \otimes \mu_\sigma,$$

where we use the standard notations and the right hand side has the algebra structure defined in (1).

(3) Let $L'/L/k$ be as above. Conclude that in $\mathrm{Br}(k)$ we have

$$[A(\varphi)] = [A(\mathrm{Inf}_{G(L/k)}^{G(L'/k)}(\varphi))],$$

for all $\varphi \in H^2(G(L/k), L^\times)$.

(4) Let k^{sep} be a separable closure of k and define

$$H^2(G(k^{\mathrm{sep}}/k), k^{\mathrm{sep}\times}) := \varinjlim_{L/k} H^2(G(L/k), L^\times),$$

where the limit is over all finite Galois extensions L/k and the transition maps are given by the inflation maps. Show that there is an isomorphism (induced by (1))

$$H^2(G(k^{\mathrm{sep}}/k), k^{\mathrm{sep}\times}) \xrightarrow{\simeq} \mathrm{Br}(k).$$

Exercise 12.3. Let k be a field, $a \in k^\times$ and $\chi : G(k^{\mathrm{sep}}/k) \to \mathbb{Q}/\mathbb{Z}$ a continuous homomorphism of order n. Let L/k be the cyclic Galois extension and $\sigma \in G(L/k)$ be the generator corresponding to χ (see Exercise 11.2 (1)). Let $\varphi_{\chi,a} \in Z^2(G(L/k), L^\times)$ be the normalized 2-cocycle defined in Exercise 11.2.

(1) Show that the class of $\varphi_{\chi,a} \in H^2(G(L/k), L^\times)$ is equal to

$$[\varphi_{\chi,a}] = a^{(2)} \circ \delta^1(\chi),$$

where $\delta^1 : H^1(G(L/k), \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G(L/k), \mathbb{Q}/\mathbb{Z}) \to H^2(G(L/k), \mathbb{Z})$ is the map from Exercise 12.1, (4) computed using the exact sequence of trivial Galois modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$

and $a^{(2)}$ denotes the map from 12.1, (3) induces by the $G(L/k)$-equivariant homomorphism $\mathbb{Z} \to L^\times$, $n \mapsto a^n$.

(2) Conclude that in $H^2(G(k^{\rm sep}/k), k^{\rm sep\times})$ we have

$$[\varphi_{\chi+\chi',a}] = [\varphi_{\chi,a}] + [\varphi_{\chi',a}], \quad [\varphi_{\chi,aa'}] = [\varphi_{\chi,a}] + [\varphi_{\chi,a'}],$$

for all continuous homomorphisms $\chi, \chi' : G(k^{\rm sep}/k) \to \mathbb{Q}/\mathbb{Z}$ and elements $a, a' \in k$. (*Hint*: For the first equation use Exercise 12.1, (7).)

(3) Conclude that we have a bilinear homomorphism

$$\mathrm{Hom}_{\rm cont}(G(k^{\rm sep}/k), \mathbb{Q}/\mathbb{Z}) \times k^\times \to \mathrm{Br}(k), \quad (\chi, a) \mapsto [A(\varphi_{\chi,a})].$$