Exercise 12 for Number theory III

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Exercise 12.1. Let $G$ be a finite group (written multiplicatively) and $A$ an abelian group (written additively) with $G$ action

$$G \times A \to A, (g, a) \mapsto g \cdot a.$$  

Set $C^0(G, A) := A$ and $C^i(G, A) := \{\text{set maps } \varphi : \prod_{j=1}^i G \to A\}$, $i \geq 1$. The addition on $A$ induces a structure of an abelian group on $C^i(G, A)$, all $i$, and we define group homomorphisms $\partial^i : C^i(G, A) \to C^{i+1}(G, A)$, $i = 0, 1, 2$, by

$$\partial^0(a)(g) := g \cdot a - a,$$

$$\partial^1(\alpha)(g, h) := g \cdot \alpha(h) - \alpha(gh) + \alpha(g),$$

$$\partial^2(\varphi)(f, g, h) := f \cdot \varphi(g, h) - \varphi(gh) + \varphi(f, gh) - \varphi(f, g),$$

for $a \in A$, $f, g, h \in G$, $\alpha \in C^1(G, A)$, $\varphi \in C^2(G, A)$.

Set $B^0(G, A) := 0$ and $B^i(G, A) := \text{Im}(\partial^{i-1})$, $i = 1, 2$, $Z^i(G, A) := \text{Ker}(\partial^i)$, $i \in [0, 2]$.

(1) Show $\partial^{i+1} \circ \partial^i = 0$, $i = 0, 1$. In particular $B^i(G, A) \subset Z^i(G, A)$ and we set

$$H^i(G, A) := Z^i(G, A) / B^i(G, A), \quad i = 0, 1, 2.$$  

(2) Show that $H^0(G, A) = A^G$ and that if $G$ acts trivial on $A$ (i.e. $g \cdot a = a$ for all $g \in G$ and $a \in A$), then $H^1(G, A) = \text{Hom}(G, A) = \text{Hom}(G^{ab}, A)$, where $\text{Hom}(-, -)$ denotes group homomorphisms and $G^{ab}$ is the maximal abelian quotient of $G$.

(3) Let $r : A \to B$ be a $G$-equivariant group homomorphism (i.e. $r(g \cdot a + a') = g \cdot r(a) + r(a')$, $g \in G$, $a, a' \in A$). Show that the composition $C^i(G, A) \to C^i(G, B)$, $\varphi \mapsto r \circ \varphi$ induces a group homomorphism

$$r^{(i)} : H^i(G, A) \to H^i(G, B), \quad i = 0, 1, 2.$$  

\[1\]This exercise sheet will be discussed on January 23. If you have questions or remarks please contact kay.ruelling@fu-berlin.de or kindler@math.fu-berlin.de or l.zhang@fu-berlin.de
(4) Let \(0 \to A \xrightarrow{r} B \xrightarrow{s} C \to 0\) be a short exact sequence of abelian groups with \(G\) action and assume the maps \(r\) and \(s\) are \(G\)-equivariant. Show that there is a group homomorphism 
\[
\delta^i : H^i(G, C) \to H^{i+1}(G, A), \quad i = 0, 1
\]
constructed as follows:
(a) For \(\varphi \in Z^i(G, C)\) show there is an element \(\tilde{\varphi} \in C^i(G, B)\) with \(\varphi = s \circ \tilde{\varphi}\).
(b) Show that the element \(\partial^i(\tilde{\varphi}) \in C^{i+1}(G, B)\) maps to zero under \(C^{i+1}(G, B) \xrightarrow{s_0} C^{i+1}(G, C)\).
(c) Conclude that there is a unique element \(\psi \in C^{i+1}(G, A)\) with \(r \circ \psi = \partial^i(\tilde{\varphi})\).
(d) Show that \(\psi \in Z^{i+1}(G, A)\).
(e) Show that the class \(\psi\) in \(H^{i+1}(G, A)\) is independent from the choice of \(\tilde{\varphi}\) in (a).
(f) Show that this construction induces a well-defined map \(\delta^i\).

(5) Let \(H \triangleleft G\) be a normal subgroup. Notice that we get a natural \(G/H\)-action on \(A_H\) and there is a natural map \(C^i(G/H, A_H) \to C^i(G, A)\) induced by precomposing with the natural surjection \(\prod G \to \prod G/H\) and composing with the inclusion \(A^H \subset A\). Show that this induces a natural map (called the inflation map)
\[
\text{Inf} = \text{Inf}_{G/H} : H^i(G/H, A^H) \to H^i(G, A), \quad i \in [0, 2].
\]

(6) Show that if \(H' \triangleleft H \triangleleft G\) is a chain of normal subgroups, then
\[
\text{Inf}^G_{G/H} = \text{Inf}^G_{G/H'} \circ \text{Inf}^{G/H'}_{G/H}.
\]

(7) Let \(H \triangleleft G\) be a normal subgroup. Assume we have a exact sequence as in (4) and that \(H\) acts trivially on \(A, B, C\). Show that we have a commutative diagram for \(i = 0, 1\)
\[
\begin{array}{ccc}
H^i(G/H, C) & \xrightarrow{\delta^i} & H^{i+1}(G/H, A) \\
\text{Inf} & & \text{Inf} \\
\downarrow & & \downarrow \\
H^i(G, C) & \xrightarrow{\delta^i} & H^{i+1}(G, A). \\
\end{array}
\]

**Exercise 12.2.** Let \(k\) be a field and \(L/k\) a finite Galois extension with Galois group \(G = \text{Gal}(L/k)\). Recall from the lecture that we have an isomorphism \(H^2(G, L^\times) \xrightarrow{\sim} \text{Br}(L/k), [\varphi] \mapsto [A(\varphi)]\).

(1) Let \(\varphi : G \times G \to L^\times\) be a normalized 2-cocycle. Let \(\{e_\sigma\}_{\sigma \in G}\) be an \(L\)-basis of \(A(\varphi)\) satisfying \(e_\sigma \cdot \lambda = \sigma(\lambda)e_\sigma\), for \(\lambda \in L\), and \(e_\sigma \cdot e_\tau = \varphi(\sigma, \tau)e_{\sigma\tau}\). We form the left \(L\)-vector space \(A(\varphi) \otimes_L L\).
$M_m(L)$, where the tensor product is formed using the left $L$-vector space structure of $A(\varphi)$.

(a) Show that there is a unique $k$-algebra structure on $A(\varphi) \otimes_k M_m(k)$ such that

$$(e_\sigma \otimes \alpha) \cdot (e_\tau \otimes \beta) = e_\sigma e_\tau \otimes \alpha \sigma(\beta).$$

(b) Show that there is an isomorphism of $k$-algebras $A(\varphi) \otimes_k M_m(k) \cong A(\varphi) \otimes_L M_m(L)$.

(2) Let $L'/L$ be another finite Galois extension of degree $m$. Set $G' = G(L'/k)$ and denote by $G' \to G, \sigma \mapsto \bar{\sigma}$ the quotient map. Fix a basis of $L'/L$ and denote by $\mu_\sigma \in M_m(L)$ the matrix obtained by the action of $\sigma$ on this fixed basis. (Be aware that $\sigma$ is not $L$-linear!) Show that there is an isomorphism of $k$-algebras

$A(\text{Inf}_{G(L'/k)}(\varphi)) \cong A(\varphi) \otimes_L M_m(L), \quad e_\sigma \mapsto e_\bar{\sigma} \otimes \mu_\sigma,$

where we use the standard notations and the right hand side has the algebra structure defined in (1).

(3) Let $L'/L/k$ be as above. Conclude that in $\text{Br}(k)$ we have

$$[A(\varphi)] = [A(\text{Inf}_{G(L'/k)}(\varphi))],$$

for all $\varphi \in H^2(G(L/k), L^\times)$.

(4) Let $k_{\text{sep}}$ be a separable closure of $k$ and define

$$H^2(G(k_{\text{sep}}/k), k_{\text{sep}}^\times) := \varinjlim L/k H^2(G(L/k), L^\times),$$

where the limit is over all finite Galois extensions $L/k$ and the transition maps are given by the inflation maps. Show that there is an isomorphism (induced by (1))

$$H^2(G(k_{\text{sep}}/k), k_{\text{sep}}^\times) \cong \text{Br}(k).$$

**Exercise 12.3.** Let $k$ be a field, $a \in k^\times$ and $\chi : G(k_{\text{sep}}/k) \to \mathbb{Q}/\mathbb{Z}$ a continuous homomorphism of order $n$. Let $L/k$ be the cyclic Galois extension and $\sigma \in G(L/k)$ be the generator corresponding to $\chi$ (see Exercise 11.2 (1)). Let $\varphi_{\chi,a} \in Z^2(G(L/k), L^\times)$ be the normalized 2-cocycle defined in Exercise 11.2.

(1) Show that the class of $\varphi_{\chi,a} \in H^2(G(L/k), L^\times)$ is equal to

$$[\varphi_{\chi,a}] = a(2) \circ \delta^1(\chi),$$

where $\delta^1 : H^1(G(L/k), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(G(L/k), \mathbb{Q}/\mathbb{Z}) \to H^2(G(L/k), \mathbb{Z})$ is the map from Exercise [12.1] (4) computed using the exact sequence of trivial Galois modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.
and $a^{(2)}$ denotes the map from 12.1 (3) induces by the $G(L/k)$-equivariant homomorphism $\mathbb{Z} \to L^\times$, $n \mapsto a^n$.

(2) Conclude that in $H^2(G(k_{\text{sep}}/k), k_{\text{sep}}^\times)$ we have

\[
[\varphi_{\chi+\chi'}, a] = [\varphi_{\chi}, a] + [\varphi_{\chi'}, a], \quad [\varphi_{\chi, a'}] = [\varphi_{\chi}, a] + [\varphi_{\chi}, a'],
\]

for all continuous homomorphisms $\chi, \chi' : G(k_{\text{sep}}/k) \to \mathbb{Q}/\mathbb{Z}$ and elements $a, a' \in k$. (Hint: For the first equation use Exercise 12.1 (7).)

(3) Conclude that we have a bilinear homomorphism

\[
\text{Hom}_{\text{cont}}(G(k_{\text{sep}}/k), \mathbb{Q}/\mathbb{Z}) \times k^\times \to \text{Br}(k), \quad (\chi, a) \mapsto [A(\varphi_{\chi, a})].
\]