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Exercise 11 for Number theory III[1]

Kay Rülling

Exercise 11.1. Let L/k be a finite Galois extension with Galois group $G = G(L/k)$. Let $\varphi : G \times G \to L^\times$ be a normalized 2-cocycle, i.e. it is a map of sets satisfying

(CC) $\rho(\varphi(\sigma,\tau)) \cdot \varphi(\rho,\sigma\tau) = \varphi(\rho\sigma,\tau)\cdot\varphi(\rho,\sigma)$, for all $\rho,\sigma,\tau \in G$.

(N) $\varphi(\mathrm{id}_L,\sigma) = 1 = \varphi(\sigma,\mathrm{id}_L)$, for all $\sigma \in G$.

(1) Show that there exists a unique (up to isomorphism) k-algebra $A(\varphi)$, which contains L and whose underlying left L-vector space has a basis $\{e_\sigma\}_{\sigma\in G}$ satisfying:
- $e_\sigma \cdot e_\tau = \varphi(\sigma,\tau)e_{\sigma\tau}$, for all $\sigma,\tau \in G$.
- $e_\sigma\lambda = \sigma(\lambda)e_\sigma$, for all $\sigma \in G$, $\lambda \in L$.

(2) Show that the k-algebra $A(\varphi)$ constructed above satisfies $c_{A(\varphi)}(L) = L$.

(3) Show that the center of $A(\varphi)$ is $c_{A(\varphi)}(A(\varphi)) = k$.

(4) Let $I \subset A(\varphi)$ be a non-zero two sided ideal. Let $x = \sum_{\sigma\in G}\lambda_\sigma e_\sigma$, with $\lambda_\sigma \in L$, be an element in $I\setminus\{0\}$. Assume that the number of non-zero coefficients λ_σ is minimal (among all elements in $I\setminus\{0\}$) and that there exists a $\sigma_0 \in G$ with $\lambda_{\sigma_0} = 1$. Show that $x = e_{\sigma_0}$.

(5) Conclude from (4) that $A(\varphi)$ is simple. All together we obtain that $A(\varphi)$ is a central simple k-algebra of degree $[A : k] = [L : k]^2$.

(6) Let $\varphi' : G \times G \to L^\times$ be another normalized 2-cocycle and assume that there exists a map $\alpha : G \to L^\times$ with

$$\alpha(\sigma)\cdot\sigma(\alpha(\tau))\cdot\varphi'(\sigma,\tau) = \alpha(\sigma\tau)\cdot\varphi(\sigma,\tau),$$

for all $\sigma,\tau \in G$ (i.e. φ and φ' have the same image in $H^2(G,L^\times)$). Show that there is an isomorphism of k-algebras $A(\varphi) \xrightarrow{\simeq} A(\varphi')$ which sends (with the obvious notations) $\lambda\cdot e_\sigma$ to $\lambda\alpha(\sigma)\cdot e'_\sigma$.

[1] This exercise sheet will be discussed on January 16. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

Exercise 11.2. Let k be a field, k^s its separable closure and denote by $G(k^s/k)$ the absolute Galois group of k. We equip $\mathbb{Q}/\mathbb{Z}$ with the discrete topology.

(1) Show that the map

$$\{\chi : G(k^s/k) \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z} \,|\, \chi \text{ is surjective and continuous}\}$$
$$\to \{(L,\sigma) \,|\, L/k \text{ cyclic Gal ext of order } n, \langle\sigma\rangle = G(L/k)\}$$

given by

$$\chi \mapsto \big((k^s)^{\mathrm{Ker}(\chi)}, \chi^{-1}(\tfrac{1}{n}) \bmod \mathrm{Ker}(\chi)\big)$$

is bijective.

(2) Let $\chi : G(k^s/k) \to \mathbb{Q}/\mathbb{Z}$ be a continuous group homomorphism of order n and denote by (L,σ) the corresponding pair from (1). Take $b \in k^\times$ and define a map $\varphi_{\chi,b} : G(L/k) \times G(L/k) \to L^\times$ via

$$\varphi_{\chi,b}(\sigma^i,\sigma^j) = \begin{cases} 1, & \text{if } i+j < n, \\ b, & \text{if } i+j \geq n, \end{cases} \quad \text{for all } i,j \in \{0,1,\ldots,n-1\}.$$

Show that $\varphi_{\chi,b}$ is a normalized 2-cocycle.

(3) Explicitly describe the central simple k-algebra $A(\chi,b) := A(\varphi_{\chi,b})$ from Exercise 11.1.

(4) For a in $k^\times$ the Galois group $G(k(\sqrt{a}/k))$ has a unique generator. Therefore by (1) there corresponds a unique continuous group homomorphism $\chi_a : G(k^s/k) \to \mathbb{Q}/\mathbb{Z}$ to the extension $k(\sqrt{a})$. Show that $A(\chi_a,b)$ is isomorphic to the quaternion algebra $A(a,b;k)$ constructed in Exercise 8.4.

Exercise 11.3. Let L/k be a finite Galois extension of degree n with Galois group $G = G(L/k)$. Set $\mathrm{Br}(k)[n] = \mathrm{Ker}(n\cdot : \mathrm{Br}(k) \to \mathrm{Br}(k))$.

(1) Let $\varphi : G \times G \to L^\times$ be a 2-cocycle. Show that φ^n is a 2-coboundary. (*Hint*: Define $\alpha : G \to L^\times$ by $\alpha(\sigma) = \prod_{\rho\in G}\varphi(\sigma,\rho)$. Show $\varphi(\sigma,\tau)^n = \alpha(\sigma)\cdot\sigma(\alpha(\tau))\cdot\alpha(\sigma\tau)^{-1}$.)

(2) Show that $\mathrm{Br}(L/k) \subset \mathrm{Br}(k)[n]$. (*Hint:* In the lecture we prove $\mathrm{Br}(L/k) = H^2(G, L^\times)$.)

(3) Conclude that any element in $\mathrm{Br}(k)$ has finite order.