Exercise 10 for Number theory III$^1$

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Exercise 10.1. Let $k$ be a field and $A$ a not necessarily commutative ring with 1 and equipped with an injective ring homomorphism $k \hookrightarrow A$. Show that the following conditions are equivalent:

1. $A$ is a central simple $k$-algebra.
2. There exists a finite Galois extension $L/k$ and an isomorphism $A \otimes_k L \cong M_n(k)$, for some $n \geq 1$.

Exercise 10.2. Let $K$ be a local field (not $\mathbb{R}$, $\mathbb{C}$), denote by $v_K : K^* \to \mathbb{Z}$ the normalized discrete valuation and by $\mathcal{O}_K$, $\mathfrak{m}$, $k$ the ring of integers, the maximal ideal and the residue field, respectively. Let $D$ be a central division algebra over $K$ and write $[D : K] = n^2$.

1. For $\alpha \in D \setminus \{0\}$ set
   \[ v_D(\alpha) := \frac{1}{[K(\alpha) : K]} v_K(Nm_{K(\alpha)/K}(\alpha)), \]
   where $Nm_{K(\alpha)/K} : K(\alpha)^* \to K^*$ is the norm map. Show that this defines a valuation on $D$ extending $v_K$, i.e. a multiplicative map $v_D : D \setminus \{0\} \to \mathbb{Q}^*$ with $v_D(\alpha + \beta) \geq \min\{v_D(\alpha), v_D(\beta)\}$ and $v_{D/K} = v_K$.

2. Show that $v_D(D \setminus \{0\}) \subset \frac{1}{n} \mathbb{Z}$. We set $e := [v_D(D \setminus \{0\}) : \mathbb{Z}]$ and get $e \leq n$.

Set

$\mathcal{O}_D := \{\alpha \in D \setminus \{0\} | v_D(\alpha) \geq 0\} \cup \{0\},$

$\mathfrak{m}_D := \{\alpha \in D \setminus \{0\} | v_D(\alpha) > 0\} \cup \{0\}.$

3. Show that $\mathcal{O}_D$ is a finite free $\mathcal{O}_K$-module of rank $n^2$ and $D = \mathcal{O}_D \otimes_{\mathcal{O}_K} K$.

4. Show that $\mathfrak{m}_D \subset \mathcal{O}_D$ is a two-sided ideal and that $k_D := \mathcal{O}_D/\mathfrak{m}_D$ is a finite commutative field extension of $k$. (Hint: Use Exercises 9.3, 9.4 and that $k$ is a finite field.)

5. Set $f := [k_D : k]$. Show $f \leq n$.

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$^1$This exercise sheet will be discussed on January 9. If you have questions or remarks please contact kay.ruebling@fu-berlin.de or kindler@math.fu-berlin.de or l.zhang@fu-berlin.de
(6) Show that \( m \cdot \mathcal{O}_D = m' \).

(7) Show that \( n^2 = e f \).

(8) Conclude that \( e = n \) and \( f = n \).

**Exercise 10.3.** Let \( K, D \) be as in Exercise 10.2.

(1) Show that there exists a subfield \( L \subset D \) with \( [L : K] = n \) and such that \( L/K \) is unramified. (*Hint:* Lift a generator of \( k_D/k \) to \( D \)).

(2) Show that

\[
\text{Br}(K) = \bigcup_{L/K \text{ fin. unramf. Gal.}} \text{Br}(L/K) = \text{Br}(K^{ur}/K),
\]

where \( L/K \) runs through the finite unramified Galois extensions and \( K^{ur} \) is the maximal unramified extension of \( K \).