ERRATUM: THE GENERALIZED DE RHAM-WITT COMPLEX
OVER A FIELD
IS A COMPLEX OF ZERO-CYCLES

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At the end of the proof of [Ru07, Th. 3.6], page 148, line 4, I use a residue
theorem [Ru07, Th. 2.19] to conclude. But there is a gap as Deligne pointed out to
me, since the theorem [Ru07, Th. 2.19] is stated for smooth projective curves over
a base field $k$, whereas in the proof of [Ru07, Th. 3.6] I use it for normal projective
curves. In this erratum I show that after a slight modification of the definition of the
residue, which replaces [Ru07, Def. 2.15], together with the results [HuKu94, Th. 1,
Th. 4] the residue theorem in fact holds for regular projective curves. Furthermore
the calculation of the residue in the course of the proof of [Ru07, Th. 3.6] is adjusted.
These modifications are enough to conclude exactly as in [Ru07] and in particular
Theorem 3.6 of [Ru07] and all other results are true and remain unchanged.

I am deeply grateful to Pierre Deligne for pointing out this mistake.

Let $k$ be a field of characteristic exponent $p \neq 2$, $S$ a finite truncation set, $C$ a
regular curve over $k$ (i.e. a one dimensional regular, integral, separated scheme of
finite type over $k$), $K = k(C)$ the function field of $C$ and $P \in C$ a closed point.

The following construction is from [HuKu94], page 88.

Denote $k(K^n_P)$ by $K_n$, $n \in \mathbb{N}$. Viewing $K$ as a constant sheaf on $C$ and $\mathcal{O}_C, K_n \subset K$
as subsheaves we define $C_n = \text{Spec}(\mathcal{O}_C \cap K_n)$. Thus we obtain maps

$$C = C_0 \rightarrow C_1 \rightarrow C_2 \rightarrow \cdots.$$ 

Since $k(\mathcal{O}_C^n) \subset (\mathcal{O}_C \cap K_n) \subset \mathcal{O}_C$ it is clear that these maps induce isomorphisms of
the underlying topological spaces and that each $C_n$ is a separated integral scheme
of finite type over $k$ with function field $K_n$. Denote by $P_n$ the image of $P$ in $C_n$.
Then for almost all $n$, $C_n$ is smooth over $k$ and $k(P_n)$ is the separable closure of $k$
in $k(P)$, by [HuKu94, Th. 1, Th. 4]. Now [Ru07, Def. 2.15] must be replaced by
the following definition (we use the notation of the article).

**Definition-Proposition 1** (cf. [Ku86], 17.4.). Let $n$ be a natural number such
that for all $n' \geq n$ (in the above notation) $C_{n'}$ is smooth over $k$ and $k(P_{n'})$ is the
separable closure of $k$ in $k(P)$. We write $\kappa_n = k(P_n)$ and $K_n = k(C_n) = k(K^n_P)$.
Finally we denote by $\hat{K}_n$ the completion of $K_n$ in $P_n$. Now the choice of a local
parameter $t$ in $P_n$ determines a unique continuous isomorphism $\kappa_n((t)) \xrightarrow{\sim} \hat{K}_n$ of
fields over $k$ (this is an isomorphism over $k$, since $\kappa_n \supset k$ is separable) and we have
a natural inclusion $\iota : \hat{K}_n \hookrightarrow \kappa_n((t))$. Take $\omega \in \mathbb{W}_S\Omega^q_{K}$, then we define the residue
of $\omega$ in $P$ to be

$$\text{Res}_{P,S}(\omega) = \text{Res}_P(\omega) = \text{Tr}_{\kappa_n/k} \left( \text{Res}_{\iota,S} \left( \iota((\text{Tr}_{K/K_n}(\omega))) \right) \right) \in \mathbb{W}_S\Omega^{q-1}_{K},$$
where the $\text{Res}_{t,S}^q$ on the right hand side, is the residue on $\mathbb{W}_S \Omega^q_{K_{n+1}(t)}$ from [Ru07, Def. 2.11]. The residue is well defined, i.e. independent of the choice of the local parameter $t$ and the number $n$.

The proof is exactly the one from [Ru07, Def. 2.15], except that we have to mention, that by the choice of $n$, $K_n$ is separable over $k$ and thus by [Ku86, 5.10. Th.] $K_{n+1} = k(K_n^p) \subset K_n$ is purely inseparable of degree $p$.

[Ru07, Rem 2.16] remains unchanged, in [Ru07, Rem 2.17] write $C_n$ instead of $C^{(p^\infty)}$, [Ru07, Prop. 2.18] remains unchanged. [Ru07, Th. 2.19] now becomes

**Theorem 2.** Let $C$ be a regular projective curve over $k$ with function field $K$. Then

$$\sum_{P \in C} \text{Res}_P(\omega) = 0,$$

for all $\omega \in \mathbb{W}_S \Omega^q_K$, $q \geq 1$.

(Notice, that $\text{Res}_P(\omega) = 0$, if $\omega$ has no pole in $P$, thus the sum is finite.)

The proof remains the same, except that at the beginning we insert the following sentence: Since $\text{Res}_P(\omega)$ is non-zero for only a finite number of points we can assume by [HuKu94, Th. 1, Th. 4] and [Ru07, Rem. 2.16] that $C$ is smooth over $k$ and the points $P$ with $\text{Res}_P(\omega) \neq 0$ are étale over $k$. In the original proof a line like this appears on page 139, line (-12) to (-10), this one has to be cancelled.

We want to take the opportunity to correct a misprint. The formula on page 139, line (-3) should be

$$\text{Res}_P(\omega) = \sum_j \text{Res}_{Q_j}(\omega_j) \text{ in } \mathbb{W}_n(\bar{k})$$

where the $Q_j$’s are the preimages of $P$ in $C \times_k \bar{k}$.

Now in the proof of [Ru07, Th. 3.6] the first paragraph remains unchanged and the beginning of the second, line (-15) to line (-9) on page 146, has to be replaced with the following (we use notation of the article):

Take $P \in \nu^{-1}(y_n = 0) \cap \Sigma$ and denote $\kappa = k(P)$. Write $K$ for the function field of $\bar{C}$ and $K_i = k(K_i^p)$, $i \geq 0$. Furthermore denote $\text{Spec}(\mathcal{O}_{\bar{C}} \cap K_i)$ by $\bar{C}_i$ and let $P_i$ be the image of $P$ in $\bar{C}_i$. Choose $l \geq 0$, such that for all $l' \geq l$ $\bar{C}_{l'}$ is smooth over $k$ and $\kappa_{l'} := k(P_{l'}) \supset k$ is separable. Let $e_P$ be the ramification index of $P$ over $P_i$ and $f_P = [\kappa : \kappa_i]$ and write

$$e_P = p^r, \quad f_P = p^s, \quad [K : K_i] = p^j.$$  

(3.6.1)

(In the article we wrongly wrote an equal sign here.)

Now in the following calculation, line (-8) on page 146 to line (-8) on page 147, replace $F^j(P)$ by $P_j$, $K_j$ by $K_i$ and $\kappa_j$ by $\kappa_i$, but the $j$’s appearing in the powers of $p$ (such as $p^{j(n-1)+r}$ etc.) stay the same. Then the whole proof of the formula in (3.6.3) goes through, since $x^{p^j} \in K_i$ for $x \in K$.

Finally at the end of the proof of [Ru07, Th. 3.6], page 148, line 4, one now refers to Theorem 2 instead of [Ru07, Th. 2.19].

The rest of the paper remains unchanged.
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References


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