Exercise 7.1. Which of the following Weierstraß equations over $\mathbb{Q}$ are smooth (and hence define elliptic curves)? Are some of them isomorphic over $\mathbb{Q}$? In the smooth cases when viewing it as an elliptic curve over $\mathbb{Q}_5$ find a minimal Weierstraß equation and say whether it has good reduction.

1. $y^2 = x^3$
2. $y^2 = x^3 + x$
3. $y^2 = x^3 + 1$
4. $y^2 = x^3 + 5x + 7$
5. $y^2 = x^3 + 20x + 56$
6. $y^2 = x^3 + 625x$

Exercise 7.2. Consider the Weierstraß equation

$$y^2 + xy + y = x^3 + x^2 + 22x - 9.$$  

Show that its discriminant is $\Delta = -2^{15} \cdot 5^2$ and $c_4 = -5 \cdot 211$. Show that if we view it as an Weierstraß equation over $\mathbb{Q}_p$, then it is minimal, where $p$ is any prime.

Exercise 7.3. Let $A$ be a DVR with residue field $k$ and fraction field $K$. Set $S = \text{Spec} A$ and denote by $s = \text{Spec} k$ and $\eta = \text{Spec} K$, the closed and generic point of $S$, respectively. Let $\pi : C \rightarrow S$ be a surjective morphism from an integral scheme $C$, such that its generic fiber $C_\eta = C \times_S \eta \rightarrow \eta$ and its special fiber $C_s = C \times_S s \rightarrow s$ are smooth of relative dimension 1. Let $\sigma : S \rightarrow C$ be a closed immersion over $S$, i.e. a section of $\pi$. Show that its ideal sheaf $\mathcal{I}$ is invertible.

(Hint: It suffices to check that the stalks in the points $\sigma(\eta)$ and $\sigma(s)$ are free of rank 1. $\mathcal{I}_{\sigma(\eta)}$ is the ideal sheaf of $\sigma(\eta) \hookrightarrow C_\eta$ and hence is invertible, since $C_\eta$ is smooth. For $\mathcal{I}_{\sigma(s)}$ it suffices to check

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1This exercise sheet will be discussed on December 9 and 16. If you have questions or remarks please contact kay.ruelling@fu-berlin.de or l.zhang@fu-berlin.de
that $\mathcal{I}_{\sigma(s)} \otimes_{\mathcal{O}_{\mathcal{C}, \sigma(s)}} k(\sigma(s))$ is $k(s)$-vector space of dimension 1 (see Exercise 5.2, (1)). To this end show that this vector space is equal to $\mathcal{O}_{\mathcal{C}, -[\sigma(s)]} \otimes_{\mathcal{O}_{\mathcal{C}, \sigma(s)}} k(s)$ to conclude. For the latter use that the sequence $0 \to \mathcal{I}_{\sigma(s)} \to \mathcal{O}_{\mathcal{C}, \sigma(s)} \to A \to 0$ is split exact and hence stays exact when pulled back to the special fiber.)

**Exercise 7.4.** Let $K$ be a complete discrete valuation field with ring of integers equal to $A$ and residue field $k$. Set $S = \text{Spec } A$ and denote by $\eta$ and $s$ the generic and the closed point of $S$, respectively. Let $E/K$ be an elliptic and $P_0 \in E(K)$ a fixed $K$-rational point. We assume $E/K$ has good reduction and denote by $\bar{E}/k$ its reduction, i.e. we take a minimal Weierstraß equation for $E/K$ which defines a model $W/A$ and the following commutative diagram in which both squares are cartesian

\[
\begin{array}{ccc}
E & \xrightarrow{j} & W \\
\downarrow & & \downarrow \\
\eta & \xrightarrow{i} & \bar{E}
\end{array}
\]

We denote by $\sigma : E(K) \to \bar{E}(k)$ the reduction map which we defined in the lecture.

1. Let $\bar{P} = (a : b : c) \in W(A) \subset \mathbb{P}^2(A)$ be an $A$-rational point (see Exercise 1.1 for notation). It defines a closed subscheme of $W$ and we denote by $\mathcal{O}_W(-[\bar{P}])$ the corresponding ideal sheaf. Show that $\mathcal{O}_W(-[\bar{P}])$ is invertible. (*Hint: Exercise 7.3*)

2. Let $\bar{P}$ be as above. Denote by $P \in E(K)$ its inverse image along $j$ and by $\bar{P}$ its inverse image along $i$. Show

\[
j^*\mathcal{O}_W(-[\bar{P}]) = \mathcal{O}_E(-[P]), \quad i^*\mathcal{O}_W(-[\bar{P}]) = \mathcal{O}_{\bar{E}}(-[\bar{P}]),
\]

where the right hand sides are the usual invertible sheaves which we defined on smooth curves.

3. Denote by $\text{Pic}^0(W)$ the subgroup of $\text{Pic}(W)$ consisting of line bundles $L$ on $W$ such that $\deg_K(j^*L) = 0$ and $\deg_k(i^*L) = 0$. Show that $j^* : \text{Pic}^0(W) \to \text{Pic}^0(E)$ is surjective. (*Hint: By the isomorphism $E(K) \cong \text{Pic}^0(E)$ any line bundle $L \in \text{Pic}^0(E)$ can be represented by a line bundle of the form $\mathcal{O}_E([P] - [P_0])$. Use this and [2]*)

4. Show that $j^* : \text{Pic}^0(W) \to \text{Pic}^0(E)$ is also injective and hence is an isomorphism.
(5) Show that the following diagram commutes
\begin{align*}
\begin{array}{ccc}
\Pic^0(W) & \xrightarrow{j^*} & \Pic^0(E) \\
\text{Pic}^0(E) & \xrightarrow{\simeq} & \Pic^0(\bar{E}) \\
E(K) & \xrightarrow{\sigma} & E(k).
\end{array}
\end{align*}

(6) Conclude that \( \sigma \) is a group homomorphism.

**Exercise 7.5.** Let \( k \) be an algebraically closed field of characteristic \( \neq 2 \). Let \( E/k \) be an elliptic curve. Show that it has a Weierstraß equation of the form
\[ y^2 = x(x-1)(x-\lambda), \quad \lambda \in k \setminus \{0,1\}. \]
This is called a Legendre equation for \( E \). (Hint: Since we are in characteristic \( \neq 2 \) we find a Weierstraß equation of the form \( y^2 = x^3 + ax^2 + bx + c \). Since \( k \) is algebraically closed we can factor the polynomial on the right of the equality as \((x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\), \( \alpha_i \in k \). Then show that the discriminant is given by \( \Delta = 16(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2 \). Since \( \Delta \neq 0 \) one can define \( \lambda = \frac{\alpha_3-\alpha_2}{\alpha_2-\alpha_1} \). Show that the Legendre form can be achieved with this particular \( \lambda \).)

**Exercise 7.6.** Let \( k \) be a field with algebraic closure \( \bar{k} \). Show that for any \( j_0 \in \bar{k} \) there exists an elliptic curve \( E/k(j_0) \) with \( j \)-invariant \( j(E) = j_0 \). Concretely, show
\begin{enumerate}
\item If \( j_0 \neq 0, 1728 \), then the Weierstraß equation
\[ y^2 + xy = x^3 - \frac{36}{j_0-1728}x - \frac{1}{j_0-1728} \]
has \( \Delta = \frac{j_0^3}{(j_0-1728)^2} \), and \( j = j_0 \).
\item The Weierstraß equation \( y^2 + y = x^3 \) has \( \Delta = -27 \) and \( j = 0 \).
\item The Weierstraß equation \( y^2 = x^3 + x \) has \( \Delta = -64 \) and \( j = 1728 \).
\end{enumerate}
The following two exercise are equal to the Exercises 6.3 and 6.4.

**Exercise 7.7.** Let \( k \) be a field.
\begin{enumerate}
\item Let \( X,Y \) be \( k \)-schemes and denote by \( p_1 : X \times Y \to X \) the projection. We have a natural map \( \Omega_{X/k}^1 \to p_{1*}\Omega_{X \times_k Y/Y}^1 \). Show the natural map induced by adjunction \( p_{1*}\Omega_{X/k}^1 \to \Omega_{X \times_k Y/Y}^1 \) is an isomorphism. (Hint: It suffices to check this locally, hence
to show $B \otimes_k \Omega^1_{A/k} \cong \Omega^1_{A\otimes_k B/B}$. This follows easily from the universal property.)

(2) Let $G$ be a group scheme over $k$. Denote by $\pi : G \to \text{Spec } k$ the structure map, by $m : G \times_k G \to G$ the group law, by $\iota : G \to G$ the inverse and by $e : \text{Spec } k \to G$ the neutral section (see Exercise sheet 4.) Consider $G \times_k G$ as a $G$-scheme via the second projection $p_2$. Show that $\tau = m \times p_2 : G \times_k G \to G \times_k G$ is an automorphism of $G$-schemes.

(3) Show that $m^* \Omega^1_{G/k} \cong p_1^* \Omega^1_{G/k}$. (Hint: From (2) we get an isomorphism $\tau^* \Omega^1_{G \times_k G/G} \cong \Omega^1_{G \times_k G/G}$. Then use (1).)

(4) Show that $\Omega^1_{G/k} \cong \pi^* e^* \Omega^1_{G/k}$. (Hint: Pullback (3) along $\text{id} \times \iota : G \to G \times_k G$.)

**Exercise 7.8.** Let $C$ be a smooth projective curve over a field $k$ which has the structure of a group scheme. Show that $C$ is an elliptic curve. (Hint: Use Exercise 7.7 (4) to show that $\omega_C$ is trivial and conclude.)