Exercise 3.1. Let $k$ be a field and $C$ a smooth projective and geometrically connected curve with function field $K$. Let $D = \sum_i n_i [x_i]$, $x_i \in C$, be a divisor on $C$ and define a presheaf $\mathcal{O}_C(D)$ on $C$ via

$$C \supset U \mapsto \mathcal{O}_C(D)(U) := \{ f \in K^\times \mid \text{div}(f)|_U \geq -D|_U \},$$

where the restriction maps are induced by the identity map on $K$. Here we use the following notation: If $E = \sum_j m_j[y_j]$ is a divisor on $C$, then we set $E|_U := \sum_j \text{with } y_j \in U m_j[y_j]$; it is a divisor on $U$. Show:

1. $\mathcal{O}_C(D)$ is a sheaf of $\mathcal{O}_C$-modules.
2. There is an open cover $C = \bigcup_j U_j$ and functions $f_j \in K^\times$ such that $D|_{U_j} = \text{div}(f_j)|_{U_j}$ and $f_i/f_j \in \mathcal{O}(U_i \cap U_j)^\times$.
3. Let $\{(U_j, f_j)\}$ be as above. Then $\mathcal{O}_C(D)|_{U_j} = \mathcal{O}_{U_j} \cdot \frac{1}{f_j}$. In particular $\mathcal{O}_C(D)$ is a locally free sheaf of rank 1.
4. Let $0_C$ be the zero-divisor. Then $\mathcal{O}_C(0_C) = \mathcal{O}_C$.
5. Let $D'$ be another divisor on $C$. Then $\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D') \cong \mathcal{O}_C(D + D')$.
6. If $D' = D + \text{div}(f)$, for some $f \in K^\times$. Then $\mathcal{O}_C(D') \cong \mathcal{O}_C(D)$.
7. $\text{Hom}_{\mathcal{O}_C}(\mathcal{O}_C(D), \mathcal{O}_C) \cong \mathcal{O}_C(-D)$.
8. Assume $D \geq 0$, i.e. $D$ is effective, i.e. $n_i \geq 0$ for all $i$. Set $D := \text{Spec} (\prod_i \mathcal{O}_{C,x_i}/m_i^{n_i})$, where the $m_i \subset \mathcal{O}_{C,x_i}$ is the maximal ideal. Then we can define a closed immersion $i : D \hookrightarrow C$ such that the following sequence is exact

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \xrightarrow{i^*} i_*\mathcal{O}_D \to 0.$$

$D$ is called the subscheme associated to $D$ and is often simply denoted by $D$ again.
9. Assume $\deg(D) := \sum_i n_i [k(x_i) : k] < 0$. Then $\Gamma(C, \mathcal{O}_C(D)) = \{0\}$. (Hint: We will prove in the lecture that $\deg(\text{div}(f)) = 0$. You can use it.)
Recall: Let $X$ be a noetherian integral scheme with function field $K$. Denote by $X^{(1)}$ the set of all points $x \in X$ of codimension 1, i.e. the closure $\overline{x}$ of $x$ in $X$ has codimension 1. We assume that for all $x \in X^{(1)}$ the local ring $\mathcal{O}_{X,x}$ is a DVR (e.g. $X$ normal or smooth over a field); we denote by $v_x : K^\times \to \mathbb{Z}$ the corresponding normalized discrete valuation. Then by definition

$$\text{CH}^1(X) := \text{coker}(K^\times \xrightarrow{\text{div}} \bigoplus_{x \in X^{(1)}} \mathbb{Z} \cdot \overline{x}),$$

where $\text{div}(f) = \sum_{x \in X^{(1)}} v_x(f) \cdot \overline{x}$ (it is a finite sum as we saw in the lecture).

**Exercise 3.2.** Let $k$ be a field. Show:

1. If $X = \text{Spec } A$ and $A$ is a unique factorization domain, then $\text{CH}^1(X) = 0$. In particular $\text{CH}^1(\mathbb{A}^n_k) = 0$.
2. Let $H \subset \mathbb{P}^n_k$ be a hyperplane (i.e. given by the vanishing of a linear homogenous polynomial in $k[x_0, \ldots, x_n]$). Then the map $\mathbb{Z} \to \text{CH}^1(\mathbb{P}^n_k), \ d \mapsto \text{class of } d \cdot H$, is an isomorphism.

**Exercise 3.3.** Let $C$ be a smooth projective curve over a field $k$ with function field $K$. Let $f \in K$ be a function.

1. Show that there is a unique $k$-morphism $\varphi_f : C \to \mathbb{P}^1_k$ such that on any open affine $U = \text{Spec } A \subset C$ on which $f$ is regular (i.e. $f \in A$) the restriction $\varphi_f|_U$ factors as $U \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$, where $U \to \mathbb{A}^1_k$ is induced by $k[t] \to A, t \mapsto f$.
2. Show that the image of $\varphi_f$ is a point if and only if $f \in K$ is algebraic over $k$.
3. Show that $\varphi_f$ is dominant (i.e. $\varphi_f$ maps the generic point on $C$ to the generic point on $\mathbb{P}^1_k$) if and only if $f$ is transcendental over $k$.
4. Assume $f$ is transcendental over $k$. Show that $\varphi_f$ is finite and surjective. ($\text{Hint: }$ We proved the finiteness in the lecture.)
5. Assume $f$ is transcendental over $k$. There are unique effective divisors $\text{div}_+(f)$, $\text{div}_-(f) \geq 0$ on $C$ such that $\text{div}(f) = \text{div}_+(f) - \text{div}_-(f)$. Set $n := \text{deg}(\text{div}_+(f))$. Show that $n \geq 1$ and that the field extension $k(t) = k(\mathbb{P}^1_k) \hookrightarrow K$ induced by $\varphi_f$ has degree $[K : k(t)] = n$. ($\text{Hint: }$ By 4 above $\varphi_f^{-1}(\mathbb{A}^1) = \text{Spec } B$ with $B$ finite over $k[t]$. Then $B$ is a free $k[t]$-module of rank $= \dim_k B/(f)$.)
6. Conclude that if there exists a function $f \in K$ with $\text{deg}(\text{div}_+(f)) = 1$, then $\varphi_f : C \to \mathbb{P}^1_k$ is an isomorphism.