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① Exterior product

A ring, $M = A\text{-mod}$.

Consider $T(M) := \bigoplus_{i \geq 0} T^i(M)$

with $T^i(M) := M \otimes_{A \otimes \dots \otimes A} M$
 $\underbrace{}_{i \text{-times}}$

as graded (non-commutative) ring via

$$T^i(M) \times T^j(M) \rightarrow T^{i+j}(M)$$

$$(m_1 \otimes \dots \otimes m_i, n_1 \otimes \dots \otimes n_j) \mapsto m_1 \otimes \dots \otimes m_i \otimes n_1 \otimes \dots \otimes n_j$$

Let $\mathcal{J} \subset T(M)$ be the two-sided homogeneous ideal

generated by $m \otimes m \in T^2(M)$, $m \in M$

set $\Lambda(M) := \frac{T(M)}{\mathcal{J}} = \text{Exterior algebra}$
 of the $A\text{-mod } M$

(2) Prop A, M as above

i) $\Lambda(M) = \bigoplus_{i \geq 0} \Lambda^i(M)$ is a graded
(non-commutative
ring)

where $\Lambda^0(M) = A$

$\Lambda^1(M) = M$

ii) Denote the image of $m_1 \otimes \dots \otimes m_r \in T(M)$
in $\Lambda(M)$ by $m_1 \wedge \dots \wedge m_r$.

Then $\Lambda^i(M)$ is as A -mod generated by

$m_1 \wedge \dots \wedge m_r$

and we have

- if $m_1 \wedge \dots \wedge m_r = 0$ if $m_{i_0} = m_{i_1}$ for some $i_0 \neq i_1 \in \{1, \dots, r\}$
- ii) $m_1 \wedge \dots \wedge m_{i_0} \wedge m_{i_1+1} \wedge \dots \wedge m_r = -m_1 \wedge \dots \wedge m_{i_0+1} \wedge m_{i_1} \wedge \dots \wedge m_r$
- iii) $(am_1 + bm_2) \wedge m_2 \wedge \dots \wedge m_r = m_1 \wedge m_2 \wedge \dots \wedge m_r + am_1 \wedge m_2 \wedge \dots \wedge m_r$.

$$3) m_i = \sum_{j=1}^r a_{ij} m_j \quad i = 1, \dots, r$$

$$\Rightarrow m_1 \wedge \dots \wedge m_r = \det(a_{ij}) \quad m_1 \wedge \dots \wedge m_r$$

$$4) \text{Hom}_A(\Lambda^i M, N) = \text{Alt}^i(M, N)$$

$$= \{ \varphi: \underbrace{M \times \dots \times M}_{i-\text{times}} \rightarrow N \mid \begin{array}{l} \text{A-multilinear and} \\ \text{alternating} \end{array} \}$$

i.e. $\varphi(m_1, \dots, m_i, m_i, \dots, m_r) = 0$

5) M free of rank n with basis
 e_1, \dots, e_n

$\Rightarrow \Lambda^i M$ free A -mod with
basis $\{e_{j_1} \wedge \dots \wedge e_{j_i}\} \quad 1 \leq j_1 < \dots < j_i \leq n$

6) $M \xrightarrow{\varphi} M' \rightarrow \Lambda(M) \xrightarrow{\Lambda(\varphi)} \Lambda(M')$
gradable form

φ is surj $\Rightarrow \Lambda(\varphi)$ surj:

Pf: $\forall \checkmark$

2) Note $m_1 m = 0$ by defn and iii) by defn.

$$\begin{aligned} \Rightarrow 0 &= (m_1 + m_2) \wedge (m_1 + m_2) \\ &= m_1 \wedge m_2 + m_2 \wedge m_1 \end{aligned}$$

\Rightarrow ii), i)

3) \Leftarrow 2), 4) by defn

5) \Leftarrow 4), 6) is clear.

③ Prop X scheme, \mathcal{F} \mathcal{O}_X -mod

define

$\Lambda^i \mathcal{F}$ as sheafification of

$$U \mapsto \bigwedge_{\mathcal{O}(U)}^i \mathcal{F}(U)$$

1) Have $\mathcal{F}^{\otimes_{\mathcal{O}_X} i} \rightarrow \Lambda^i \mathcal{F}$

2) $\mathcal{F} \in Qcoh(X) \Rightarrow \Lambda^i \mathcal{F} \in Qcoh(X)$

3) X noeth, $\mathcal{F} \in \mathcal{O}h(X) \Rightarrow \Lambda^i(\mathcal{F}) \in \mathcal{O}h(X)$

4) \mathcal{F} locally free finite rank n

$\Rightarrow \Lambda^i \mathcal{F}$ locally free of rank $\binom{n}{i}$, $i \leq n$

and $\Lambda^i \mathcal{F} = 0 \vee i > n+1$

5) $\Lambda^0 \mathcal{F} = \mathcal{O}_X$, $\Lambda^n \mathcal{F} = \mathcal{F}$

Pf: 1) By construction

1) \Rightarrow 2), 3), 2) + (2), 5) \Rightarrow 4)

5) \vee

□

④ Prop. X scheme

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

s.e.s of locally free sheaves on X
of finite rank.

Then $\wedge^n E$ \cong \exists filtration

$$\wedge^r E = \overline{F}^0 > \overline{F}^1 > \dots > \overline{F}^r > \overline{F}^{r+1} = 0$$

s.t.

$$\frac{\overline{F}^j}{\overline{F}^{j+1}} \cong \wedge^j(E') \otimes_{\mathcal{O}_X} \wedge^{r-j}(E'')$$

In part if $\text{rk}(E) = n$
 $\text{rk}(E') = n'$, $\text{rk}(E'') = n''$

$$\Rightarrow n = n' + n'' \text{ and } \wedge^n(E) \cong \wedge^{n'}(E') \otimes_{\mathcal{O}_X} \wedge^{n''}(E'')$$

isom of line bundles

Pf: "In part" follows from statement

$$\text{and } \wedge^j(E') = 0 \quad \forall j \geq n'+1, \quad \wedge^{n-j}(E'') = 0 \quad \forall j \leq n'-1$$

define

$$F^j = \text{Im} (\Lambda^j(E') \otimes \Lambda^{r-j} E \rightarrow \Lambda^r E)$$

$\epsilon_1 \wedge \dots \wedge e_j \otimes f_1 \wedge \dots \wedge f_{r-j} \mapsto \epsilon_1 \wedge \dots \wedge e_j \wedge f_1 \wedge \dots \wedge f_{r-j}$

$1 \leq j \leq r$

$$\Rightarrow \Lambda^r E =: F^0 \supset F^1 \supset \dots \supset F^r \supset F^{r+1} =: 0$$

define

$$\Lambda^j E' \otimes \Lambda^{r-j} E'' \xrightarrow{\quad \text{O}_X \quad} \frac{F^j}{F^{j+1}} \quad (*)$$

$(\epsilon_1 \wedge \dots \wedge e_j) \otimes (f_1 \wedge \dots \wedge f_{r-j}) \mapsto \overline{(\epsilon_1 \wedge \dots \wedge e_j \wedge \tilde{f}_1 \wedge \dots \wedge \tilde{f}_{r-j})}$

choose $\tilde{f}_i \in E$

lifts of f_i
(locally)

(well defined \rightarrow if \tilde{f}_i is a different lift
of $f_i \Rightarrow \tilde{f}_i = \tilde{f}'_i + \epsilon, \epsilon \in E'$)

check (*) isom \rightarrow wlog $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$
free and split

e_1, \dots, e_m basis of E' , $e_1, \dots, e_m, e_{m+1}, \dots, e_{m+n}$ Basis of E

$\Rightarrow \{e_{i_1} \wedge \dots \wedge e_{i_j} \otimes e_{\sigma_1} \wedge \dots \wedge e_{\sigma_{r-j}} \mid \begin{array}{l} 1 \leq i_1 \leq i_2 \leq \dots \leq m, 1 \leq \sigma_1 < \dots < \sigma_{r-j} \leq m+n \\ \{i_1, \dots, i_j\} \cap \{\sigma_1, \dots, \sigma_{r-j}\} = \emptyset \end{array}\}$ basis of F^j

This implies the statement. \square

(5) Def. $X \rightarrow Y$ scheme morph.

set $\Omega_{X/Y}^n := \bigwedge^n \Omega_{X/Y}^1$

$$\left(\Omega_{X/Y}^0 = \mathcal{O}_X \right)$$

(6) Remark One can show that

$$d: \mathcal{O}_X \rightarrow \Omega_{X/Y}^1, a \mapsto da$$

extends to

$$d: \Omega_{X/Y}^n \rightarrow \Omega_{X/Y}^{n+1} \quad \text{well-def}$$

$$a db_1 \wedge \dots \wedge db_n \mapsto da \wedge db_1 \wedge \dots \wedge db_n$$

$$\text{and } d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^i \alpha \wedge d\beta$$

$$\alpha \in \Omega_{X/Y}^i, \beta \in \Omega_{X/Y}^j$$

$$\text{and } d \circ d = 0$$

$$\rightarrow \text{get de Rham complex } \Omega_{X/Y}: \mathcal{O}_X \rightarrow \Omega_{X/Y}^1 \xrightarrow{d} \Omega_{X/Y}^2 \xrightarrow{d} \dots$$

(7) Gs.

R ring

$$\bar{P} = \bar{\mathbb{P}}_R^n$$

$$\Rightarrow \mathcal{I}_{\bar{P}/R}^n \cong \mathcal{O}_P(-n-1)$$

and on $\mathcal{U}_i = \{x_i \neq 0\}$ this isom is given by

$$d\left(\frac{x_0}{x_i}\right) \wedge \dots \wedge \widehat{d\left(\frac{x_i}{x_i}\right)} \wedge \dots \wedge d\left(\frac{x_n}{x_i}\right) \mapsto (-1)^{n-i} \frac{1}{x_i^{n+1}}$$

Pf. Euler seq

$$0 \rightarrow \mathcal{I}_{\bar{P}/R}^1 \rightarrow \bigoplus_{i=0}^n \mathcal{O}_P(-1) \rightarrow \mathcal{O}_P \rightarrow 0$$

$$\Rightarrow \bigwedge^{n+1} \left(\bigoplus_{i=0}^n \mathcal{O}_P(-1) \right) = \bigwedge^{n+1} \mathcal{I}_{\bar{P}/R}^1 \otimes \bigwedge^{n+1} \mathcal{O}_P = \mathcal{I}_{\bar{P}/R}^n$$

$$\begin{aligned} & \text{inductively} \\ & (\mathcal{O}_P(-1))^{\otimes n+1} \\ & \quad \text{"/} \\ & \mathcal{O}_P(-n-1) \end{aligned}$$

For the local description (assume $i=0$)

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$$S_0 = R[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}], \quad \mathcal{U}_0 = \text{Spec } S_0$$

$$\bigoplus_{i=0}^n T(\mathcal{U}_0, \mathcal{O}(-1)) = \bigoplus_{i=0}^n \frac{1}{x_0} S_0 e_i$$

Euler Eq on \mathcal{U}_0
(see pf of §10(④))

$$0 \rightarrow \sum_{i=0}^1 S_0 e_i \rightarrow \bigoplus_{i=1}^n \left(\frac{1}{x_0} e_i - \frac{x_i}{x_0^2} e_0 \right) S_0 \oplus \frac{1}{x_0} e_0 S_0 \xrightarrow{\sigma} S_0 \rightarrow 0$$

$$d\left(\frac{x_i}{x_0}\right) \mapsto \left(\frac{1}{x_0} e_i - \frac{x_i}{x_0^2} e_0 \right)$$

$\begin{cases} x_i \\ x_0 \end{cases}$

The isom

$$\Lambda^n \left(\sum_{i=0}^1 S_0 e_i \right) \otimes_{S_0} S_0 \rightarrow \Lambda^{n+1} \left(\bigoplus_{i=0}^n \frac{1}{x_0} e_i \right)$$

is by the proof of ④ induced by

$$d\left(\frac{x_1}{x_0}\right) \wedge \dots \wedge d\left(\frac{x_n}{x_0}\right) \otimes 1 \mapsto \left(\frac{1}{x_0} e_1 - \frac{x_1}{x_0^2} e_0 \right) \wedge \dots \wedge \left(\frac{1}{x_0} e_n - \frac{x_n}{x_0^2} e_0 \right) \wedge \frac{1}{x_0} e_0$$

$$= \frac{1}{x_0^{n+1}} e_1 \wedge \dots \wedge e_n \wedge e_0$$

$$= (-1)^n \frac{1}{x_0^{n+1}} e_0 \wedge \dots \wedge e_n \quad \square$$

Ex. Check it glued $n=2$ on $U_0 \cap U_1$

$$d\left(\frac{x_1}{x_0}\right) \wedge d\left(\frac{x_2}{x_0}\right) \rightarrow (-1)^2 \frac{1}{x_0^3}$$

||

$$d\left(\frac{x_0}{x_1}\right)^{-1} \wedge d\left(\frac{x_2}{x_1}\right)$$

||

$$\begin{aligned} & -\left(\frac{x_0}{x_1}\right)^{-2} d\left(\frac{x_0}{x_1}\right) \wedge \left(\frac{x_1}{x_0}\right) d\left(\frac{x_2}{x_1}\right) \\ & = -\left(\frac{x_1}{x_0}\right)^3 d\left(\frac{x_0}{x_1}\right) d\left(\frac{x_2}{x_1}\right) \mapsto -\left(\frac{x_1}{x_0}\right)^3 (-1)^{2-1} \frac{1}{x_1^3} \quad \text{OK} \end{aligned}$$

(7) Cor.

$X \xrightarrow{i} P$
 rel dim \sum_n $y \xleftarrow{\text{sum, rel } n+r}$ with ideal sheaf I
 codim r

$$\Rightarrow i^* \Omega_{P/Y}^{n+r} \cong \wedge^r (I/I^2) \otimes \Omega_{X/Y}^r$$

rk: § 11, (8) and (4) \square