§ 7 Cohomology for quasi coherent sheaves

(1) Recollection on quasi-coherent sheaves

A ring, $M = A$-module, $X = \operatorname{Spec} A$

Denote by $\tilde{M}$ the sheaf on X associated to the presheaf

$$U \supset V \longmapsto M \otimes_A \mathcal{O}_X(V)$$

We say $\tilde{M}$ is the sheaf assoc. to M on X

We have the following properties

(see e.g. Hartshorne, II, § 5)

(i) $\tilde{M}$ is a sheaf of $\mathcal{O}_U$-mod

and $\mathcal{O}_U = \tilde{A}$

(iii) $(\tilde{M})_{\mathfrak{p}} = M_{\mathfrak{p}}$

stalk

(iii) $\Gamma(\operatorname{Spec} A_f, \tilde{M}) = M_f$

in part $\Gamma(U, \tilde{M}) = M$

(IV) $(A\text{-mod}) \longrightarrow (\mathcal{O}_{\operatorname{Spec} A}\text{-mod})$

$M \longmapsto \tilde{M}$ is an exact

fully faithful functor

i.e.

$\operatorname{Hom}_{\mathcal{O}_U}(\tilde{M}, \tilde{N}) = \operatorname{Hom}_A(M, N)$

(V) $\tilde{M} \otimes_{\mathcal{O}_U} \tilde{N} = (M \otimes_A N)^{\sim}$

(vi) $f : \operatorname{Spec} A \to \operatorname{Spec} B \Rightarrow f_* \tilde{M} = \widetilde{{}_B M}$ M as B-module

Let X = scheme, $\mathcal{F} = \mathcal{O}_X$-mod

We say $\mathcal{F}$ is <u>quasi-coherent</u>

$\Leftrightarrow \exists\ X = \bigcup U_i$ open cover

with $U_i = \operatorname{Spec} A_i$

and $\exists\ A_i$-mod M_i s.t.

$\mathcal{F}_{|U_i} \cong \widetilde{M_i}$

(2) Prop.. X scheme

(i) $\mathcal{F} \in \mathcal{SR}(X, \mathcal{O}_X)$, $U = \operatorname{Spec} A \subset X$ open

$\Rightarrow \exists$ nat map, functorial in $\mathcal{F}$

(*) $\Gamma(U, \mathcal{F})^{\sim} \longrightarrow \mathcal{F}_{|U}$

(ii) $\mathcal{F}$ is quasi-coherent $\Leftrightarrow$ (*) is an isom $\forall\ U = \operatorname{Spec} A$

(iii) The full subcategory of quasi-coh sheaves in $\mathcal{SR}(X, \mathcal{O}_X)$ is abelian.

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(iv) $\mathcal{F}, \mathcal{G}$ quasi-coh $\Rightarrow$ $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ quasi-coh

$\mathcal{F}_i$ quasi-coh, $i \in I \Rightarrow \bigoplus_{i \in I} \mathcal{F}_i$ quasi-coh

(v) $f : X \to Y$, $\mathcal{G}$ quasi-coh on Y

$\Rightarrow f^* \mathcal{G}$ quasi-coh

(vi) $f : X \to Y$, $\mathcal{F}$ quasi-coh on X

- quasi-compact, i.e. $\forall\ V = \operatorname{Spec} A \subset Y$ open affine $f^{-1}(V)$ is quasi-compact.

- quasi-separated, i.e. the diagonal $\Delta : X \to X \times_Y X$ is quasi-compact

$\Rightarrow f_* \mathcal{F}$ quasi-coh.

(e.g. X, Y noetherian)

Pf: (i) Have natural map $V \subset U$

$$\mathcal{F}(U) \otimes_{\mathcal{O}(U)} \mathcal{O}(V) \to \mathcal{F}(V)$$

$\rightsquigarrow$ map of presheaves

$\rightsquigarrow \widetilde{\mathcal{F}(U)} \to \mathcal{F}|_U$ map of sheaves.

(iii) see e.g. Hartshorne, II, Prop 5.4.

(iii) - (v) The question is whether the map from (i) is an isom (for kernels, cokernels, $f^*\mathcal{G}$, $\otimes$, ...)

→ local question

→ follows from (1)

(vi) Question is local on Y

→ wlog $Y = \operatorname{Spec} B$

f quasi-compact

$\Rightarrow \exists \quad X = f^{-1}(Y) = \bigcup_{i=1}^{r} U_i$ open affine cover

$U_i = \operatorname{Spec} A_i$

f quasi-sep $\Rightarrow$

$$\begin{array}{ccc} U_{ij} & \longrightarrow & U_i \times_Y U_j \leftarrow \text{open affine} \\ \downarrow & \ulcorner & \downarrow \\ X & \xrightarrow{\Delta} & X \times_Y X \\ & & \text{quasi-compact} \end{array}$$

$$\Rightarrow \quad U_{ij} = \bigcup_{k=1}^{n_{ij}} U_{ijk} \quad \text{with } U_{ijk} = \operatorname{Spec} A_{ijk}$$

$\alpha_i : U_i \hookrightarrow X$, $\alpha_{ijk} : U_{ijk} \hookrightarrow X$

$\mathcal{F}$ sheaf $\Rightarrow$

$$\mathcal{F} = \operatorname{Ker}\left(\prod_{i=1}^{r} \alpha_{i*}(\mathcal{F}|_{U_i}) \longrightarrow \prod_{i,j} \prod_{k=1}^{n_{ij}} \alpha_{ijk*}(\mathcal{F}|_{U_{ijk}}) \right)$$

$$= \operatorname{Ker}\Big(\underbrace{\bigoplus_{i=1}^{r} \alpha_{i*}(\mathcal{F}|_{U_i})}_{=:\mathcal{F}'} \longrightarrow \underbrace{\bigoplus_{i,j} \bigoplus_{k=1}^{n_{ij}} \alpha_{ijk*}(\mathcal{F}|_{U_{ijk}})}_{=:\mathcal{F}''} \Big)$$

$\mathcal{F}$ quasi-coh $\Rightarrow$ $\mathcal{F}'$ and $\mathcal{F}''$ quasi-coh

by (iv), (v) and (1), (vi)

and $f_* \mathcal{F}' = \bigoplus_{i=1}^{r} (f \circ \alpha_i)_* (\mathcal{F}|_{U_i})$

with $f \circ \alpha_i : U_i \longrightarrow Y = \operatorname{Spec} B$

$\underset{(1)(vi)}{\Rightarrow}$ $f_* \mathcal{F}'$ quasi-coh

similar $f_* \mathcal{F}''$ quasi-coh

and

$0 \to f_* \mathcal{F} \to f_* \mathcal{F}' \to f_* \mathcal{F}''$ exact

$\underset{(iii)}{\Rightarrow}$ $f_* \mathcal{F}$ quasi-coh $\square$

(3) Ex:

i) $\mathcal{F}$ fin pres, $\mathcal{G}$ quasi-coh

$\Rightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ quasi-coh

But in general if $\mathcal{F}, \mathcal{G}$ are quasi-coh

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ does not need to be quasi-coh.

ii) $\mathcal{F}_i$ q.c. $\not\Rightarrow \prod_i \mathcal{F}_i$ q.c.

i.A.

iii) $j : \mathcal{U} \hookrightarrow X$ open, $\mathcal{F}$ q.c. on $\mathcal{U}$

$\Rightarrow j_! \mathcal{F}$ is not q.c

(4) Lemma: $X = \operatorname{Spec} R$, $\mathcal{F}$ quasi-coh on X

$$\mathcal{U} = \{ U_i = \operatorname{Spec} R_{f_i} \}_{i=0,\dots,n}, \quad f_i \in R,$$

open cover of X

Then:

i) $\check{H}^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$

ii) $\check{H}^j(\mathcal{U}, \mathcal{F}) = 0 \quad \forall\, j \geq 1$

Pf: $M := \mathcal{F}(X, \mathcal{F})$, i.e. $\mathcal{F} = \tilde{M}$

$$C(\mathcal{U}, \mathcal{F}) = \bigoplus_{i=0}^{n} \mathcal{F}(U_i) \to \bigoplus_{0 \leq i < j \leq n} \mathcal{F}(U_i \cap U_j) \to \dots \to \mathcal{F}(U_0 \cap \dots \cap U_n) \to 0$$

Note $\mathcal{F}(U_{i_0 \dots i_r}) = M_{f_{i_0} \dots f_{i_r}}$

→ to show

$$0 \to M \to \bigoplus_{i=0}^{n} M_{f_i} \xrightarrow{\delta} \bigoplus_{0 \leq i < j \leq n} M_{f_i f_j} \to \dots \to M_{f_0 \dots f_n} \to 0$$

is exact

But $X = \hat{\bigcup_{i=0}} U_i \iff (f_0, \dots, f_n) = R$

$N = R\text{-Mod}$

$\Rightarrow N \longrightarrow \hat{\bigoplus_{i=0}^{n}} N_{f_i}$ is inj.

$(n \longmapsto 0 \iff \forall i \; \exists r_i \text{ s.t. } f_i^{r_i} n$

$\Rightarrow 1 \cdot n = \{ q_i f_i^{r_i} n = 0 \}$

$\uparrow$

$\exists q_i$

Thus $H^i(L) = 0 \iff H^i(L^\cdot) \underset{R}{\otimes} R_{f_i} = 0 \quad \forall i = 0, \dots, n$

$\parallel$ ↖ localization is exact

$H^i(L^\cdot \underset{R}{\otimes} R_{f_i})$

and

$$L^\cdot \underset{R}{\otimes} R_{f_i} = \left[F(U_i) \longrightarrow C^\cdot(\mathfrak{U}, U_i, F(U_i)) \right\}$$

§6, (5) [if $\exists U_x \in \mathfrak{U} : U_x = X$

$\Rightarrow F(X) \xrightarrow{\varepsilon} C^\cdot(\mathfrak{U}, F)$ homotopy eq

$\Rightarrow H^i(C^\cdot(\mathfrak{U}, F)) = \begin{cases} F(X), & i = 0 \\ 0, & \text{else} \end{cases}$]

$\Rightarrow H^i\left(L^\cdot \underset{R}{\otimes} R_{f_i} \right) = 0 \quad \forall i \geq 0 \quad \forall i \quad \Rightarrow H^i(L^\cdot) = 0 \; \forall i$

i.e. $L^\cdot$ exact □

5 Thm (Serre)

$X = \operatorname{Spec} R$, $\mathcal{F}$ quasi-coh

$\Rightarrow H^j(X, \mathcal{F}) = 0 \quad \forall j \geq 1$

Pf: [Recall §6, lemma 9: $(X, \mathcal{A})$ top ringed space

$\mathcal{B}$ base of top s.t. $V \in \mathcal{B} \Rightarrow V$ quasi-compact

$\emptyset \in \mathcal{B}$ and $V, U \in \mathcal{B} \Rightarrow U \cap V \in \mathcal{B}$

Let $\mathcal{D} \subset Sh(X, \mathcal{A})$ be the full subcategory consisting of $\mathcal{F} \in Sh(X, \mathcal{A})$ s.t.

$\forall\, U \in \mathcal{B}$ and all $\mathfrak{U} = (U_i)_{i=1,\dots,r}$ finite open cover of U $(U = \bigcup_i U_i)$ with $U_i \in \mathcal{B}$

we have $\check{H}^n(\mathfrak{U}, \mathcal{F}|_U) = 0 \quad \forall n \geq 1$

Then $H^n(U, \mathcal{F}|_U) = 0 \quad \forall\, \mathcal{F} \in \mathcal{D}$ and all $U \in \mathcal{B}$]

Here $(X, A) \hookrightarrow (X, \mathcal{O}_X)$ with $X = \operatorname{Spec} R$

$$\mathcal{B} = \{ \operatorname{Spec} R_f \mid f \in R \}$$

$$(\operatorname{Spec} R_f \cap \operatorname{Spec} R_g = \operatorname{Spec} R_{fg})$$

$\mathcal{F}$ quasi-coh $\overset{④}{\Rightarrow}$ $\check{H}^j(\mathcal{U}, \mathcal{F}) = 0$

$\forall$ finite open covers $\mathcal{U} \subset \mathcal{B}$

i.e. $\mathcal{F} \in \mathcal{D}$

$\overset{§6,⑨}{\Rightarrow}$ $H^j(\operatorname{Spec} R_f, \mathcal{F}|_{\operatorname{Spec} R_f}) = 0$ $\forall j \geq 1$

$\forall f \in R$

import $f = 1$ □

⑥ Recall $f : X \to Y$ is called affine

$\Longleftrightarrow$ $\forall$ $V = \operatorname{Spec} A \subset Y$ affine open

$f^{-1}(V) \subset X$ is affine

$\Leftarrow$ $\exists$ an open cover $Y = \bigcup V_i$ with

V_i affine s.t $f^{-1}(V_i) \subset X$ is affine

(7) Cor: X, Y schemes, $\mathcal{F}$ quasi-coh on X

$f: X \to Y$ affine morphism

$$\Rightarrow \quad R^i f_* \mathcal{F} = 0 \quad \forall\ i \geq 1$$

and $H^i(X, \mathcal{F}) = H^i(Y, f_* \mathcal{F}) \quad \forall\ i \geq 0$

Pf: $V \subset Y$ affine $\Rightarrow f^{-1}(V)$ affine

$$\Rightarrow H^i(f^{-1}(V), \mathcal{F}) = 0 \quad \forall\ i \geq 1$$

$\Rightarrow$ (5)

$\Rightarrow R^i f_* \mathcal{F}$ = sheaf assoc to

$$V \mapsto H^i(f^{-1}(V), \mathcal{F})$$

$$= 0$$

last assertion follows from §4, (13) $\square$

(8) Cor X scheme

Assume the diagonal

$\Delta : X \to X \times X$ is affine

(e.g. X is separated : closed immersions are affine)

$U = \operatorname{Spec} A \xrightarrow{j} X$ affine open.

$\mathcal{F}$ quasi-coh on U

$\Rightarrow R^i j_* \mathcal{F} = 0 \quad \forall i > 0$

or $H^i(X, j_* \mathcal{F}) = 0$

Pf:

j is an affine morphism:

$V = \operatorname{Spec} B \subset X$ open

$$\begin{array}{ccc} j^{-1}(V) = V \cap U & \longrightarrow & V \times U \text{ affine} \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

$\Rightarrow V \cap U$ affine

$\Rightarrow R^i j_* \mathcal{F} = 0 \Rightarrow H^i(X, j_* \mathcal{F}) = H^i(U, \mathcal{F}) = 0 \quad \forall i > 0$

§4, (13) (5) ∎

(9) Thm: X scheme s.t.

the diagonal $\Delta : X \longrightarrow X \times X$ is affine

(e.g. X is separated)

$\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering of X

with $U_i \subset X$ affine $\forall i \in I$

(I ordered set)

$\mathcal{F}$ quasi-coherent on X

Then:

$0 \to \mathcal{F} \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ (= sheaf Čech complex §6, (3))

is a resolution which

is $\Gamma(X, -)$-acyclic

(i.e. $H^r(X, \mathcal{C}^n(\mathcal{U}, \mathcal{F})) = 0 \quad \forall r \geq 1$)

In particular

$$\| \quad H^n(X, \mathcal{F}) = \check{H}^n(\mathcal{U}, \mathcal{F}) \qquad \forall n$$

$$\left(= \text{def } H^n(\Gamma(X, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}))\right)$$

Pf: $i_0, \dots, i_m \in I^{m+1}$

$\Rightarrow$ (8) + assumptions and

$$R^s j_* \left(\mathcal{F}_{|\mathcal{U}_{i_0 \dots i_m}} \right) = 0 \quad \forall\, s \geq 1$$

$$H^s \left(X, j_* \mathcal{F}_{|\mathcal{U}_{i_0 \dots i_m}} \right) = 0 \quad \forall\, s \geq 1$$

$\Rightarrow$ $j_* \mathcal{F}_{|\mathcal{U}_{i_0 \dots i_m}}$ $\Gamma(X, -)$-acyclic

$\Rightarrow$ $\mathcal{C}^m(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \dots < i_m} j_* \left(\mathcal{F}_{|\mathcal{U}_{i_0 \dots i_m}} \right)$

$\Gamma(X, -)$-acyclic

$\Rightarrow$ §6, (6) $\mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ $\Gamma(X, -)$-acyclic resolution

$\Rightarrow$ §4, (2), V)

$$\begin{aligned} H^m(X, \mathcal{F}) &= H^m \left(\Gamma(X, \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})) \right) \\ &= H^m \left(C^\bullet(\mathcal{U}, \mathcal{F}) \right) \\ &= \check{H}^m(\mathcal{U}, \mathcal{F}) \end{aligned}$$

$\square$

(10) **Thm.** $f: X \to Y$ quasi-cpt, $\mathcal{F}$ quasi-coh on X

Assume $\Delta_{X/Y}: X \to X \times_Y X$ is affine

(1) $R^n f_* \mathcal{F}$ quasi-coh $\forall\, n \geq 0$

(2) Y affine $\Rightarrow$ $R^n f_* \mathcal{F} = H^n(X, \mathcal{F})^{\sim}$

($\mathcal{O}_Y$-mod assoc to the $\mathcal{O}_Y(Y)$-mod $H^n(X, \mathcal{F})$)

(3) Let $\mathcal{U} = \{U_i\}_{i \in I}$ open affine covering of X

$\Rightarrow R^n f_* \mathcal{F} = H^n(f_* \mathcal{C}^\cdot(\mathcal{U}, \mathcal{F}))$

Pf: (3) $\varepsilon: \mathcal{F} \to \mathcal{C}^\cdot(\mathcal{U}, \mathcal{F})$ induces nat map

$H^n(f_* \mathcal{C}^\cdot(\mathcal{U}, \mathcal{F})) \to R^n f_* \mathcal{F}$ (*)

$V \subset Y$ open

$$\Rightarrow \quad H^n(f_* \mathcal{C}^\cdot(\mathcal{U}, \mathcal{F}))|_V \longrightarrow (R^n f_* \mathcal{F})|_V$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$(2^*) \quad H^n(f_{V*} \mathcal{C}^\cdot(\mathcal{U} \cap f^{-1}(V), \mathcal{F})) \longrightarrow R^n f_{V*}(\mathcal{F}|_{f^{-1}(V)})$$

where $f_V : f^{-1}(V) \to V$, $\mathcal{U} \cap f^{-1}(V)$

$$= \{ U_i \cap f^{-1}(V) \}$$

Have: $(*)$ isom $\Leftrightarrow$ (2^*) isom $\forall$ $V \subset Y$ affine

$\rightarrow$ wlog Y = affine

$\Rightarrow$ $\Delta : X \to X \times X$ affine

$\ulcorner$ since affine $\Delta_{X/Y}$

$$\begin{array}{ccc} X & \xrightarrow{\Delta} & \\ \downarrow & & \\ X \times_Y X & \hookrightarrow & X \times X \\ \downarrow & \square & \downarrow \\ Y & \xrightarrow[\text{closed}]{} & Y \times Y \end{array}$$

$\lrcorner$

$\llcorner$

and graph $\gamma_f : X \to X \times Y$ closed immersion (in part affine)

Since

$$\begin{array}{ccc} X & \xrightarrow{\gamma_f} & X \times Y \\ f \downarrow & \square & \downarrow f \times id \\ Y & \longrightarrow & Y \times Y \end{array}$$

closed immersion

Δ affine $\Rightarrow$ $j : \mathcal{U}_{i_0 \dots i_p} \longrightarrow X$ affine

$\Rightarrow$ $\mathcal{U}_{i_0 \dots i_p} \xrightarrow{\gamma_f \circ j} X \times Y$ affine

$\Rightarrow \forall\, V \subset Y$ affine : $\mathcal{U}_{i_0 \dots i_p} \cap f^{-1}(V) = (\gamma_f \circ j)^{-1}(\mathcal{U}_{i_0 \dots i_p} \times V)$ affine

$\Rightarrow$ $f \circ j : \mathcal{U}_{i_0 \dots i_p} \longrightarrow Y$ affine

Thus $\mathcal{U} \cap f^{-1}(V)$ is an affine cover of $f^{-1}(V)$

$\Rightarrow H^n(f^{-1}(V), \mathcal{F}) = \check{H}^n(\mathcal{U} \cap f^{-1}(V), \mathcal{F})$

ranging over $V \subset Y$ affine and taking sheafification yields that (2^*) is an isom.

$\Rightarrow (3)$

(2) Y affine, f quasi-cpt $\Rightarrow \exists\ \mathcal{U} = \{U_i\}_{i=0\dots N}$ finite affine open cov.

$$f_* \mathcal{C}^q(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \dots < i_p} f_* \left(j_* \mathcal{F}_{|U_{i_0 \dots i_p}} \right)$$

$f \circ j : U_{i_0 \dots i_p} \longrightarrow Y$ affine

$$\Rightarrow f_* \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) = C^\bullet(\mathcal{U}, \mathcal{F}) \widetilde{\ }$$

$\uparrow$ as $\mathcal{O}_Y$-Mod.

(3) $\Rightarrow R^q f_* \mathcal{F} = H^q(C^\bullet(\mathcal{U}, \mathcal{F})\widetilde{\ })$

$$= (\check{H}^q(\mathcal{U}, \mathcal{F}))\widetilde{\ }$$

$\uparrow$ "$\sim$" exakt

$$= (H^q(X, \mathcal{F}))\widetilde{\ }$$

(1) follows from (2) $\square$

(11) Cor (of (5)): X scheme

let

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

s.e.s of $\mathcal{O}_X$-mod.

Assume $\mathcal{F}', \mathcal{F}''$ are quasi-coherent

$\Longrightarrow$ $\mathcal{F}$ quasi-coherent

pf: $U \subset X$ open affine

(5) $\underset{\mathcal{F}' \text{ q.coh.}}{\Longrightarrow}$ $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is exact

$\underset{\text{"}\sim\text{" exact}}{\Longrightarrow}$

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\mathcal{F}'(U)} \arrow[r] \arrow[d, "\text{nat map (see (2))}"] & \widetilde{\mathcal{F}(U)} \arrow[r] \arrow[d] & \widetilde{\mathcal{F}''(U)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}'_{|U} \arrow[r] & \mathcal{F}_{|U} \arrow[r] & \mathcal{F}''_{|U} \arrow[r] & 0
\end{tikzcd}$$

commutative diagram of s.e.s

$\underset{\text{(2), (ii) snake lemma}}{\Longrightarrow}$ OK $\square$