

§6 Čech cohomology

① Čech cohomology

(X, \mathcal{A}) ringed space

$\mathcal{F} \in \mathcal{S}\mathcal{B}(X, \mathcal{A})$

$\mathcal{U} = \{U_i\}_{i \in I}$ open cover of X , with
 $(I, <)$ totally ordered set

For $i_0 < i_1 < \dots < i_n$ set $U_{i_0, \dots, i_n} := U_{i_0} \cap \dots \cap U_{i_n}$
 $i_n \in I$

We write $j_{i_0, \dots, i_n} : U_{i_0, \dots, i_n} \hookrightarrow X$

for the open immersion,

if it is clear from which U_I we are

we also just write $j = j_{i_0, \dots, i_n}$,

e.g. $j_* (\mathcal{F}|_{U_{i_0, \dots, i_n}}) = j_{i_0, \dots, i_n}^* \overset{-1}{j_{i_0, \dots, i_n}} \mathcal{F}$

define

$$\mathcal{E}^n(\mathcal{U}, \mathbb{F}) = \prod_{i_0 < \dots < i_n} d_* (\mathbb{F} | \mathcal{U}_{i_0 \dots i_n}) \in \mathcal{S}\mathcal{S}(X, A)$$

$$\tilde{C}^n(\mathcal{U}, \mathbb{F}) = \prod X, \mathcal{E}^n(\mathcal{U}, \mathbb{F}) = \prod_{i_0 < \dots < i_n} \mathbb{F}(\mathcal{U}_{i_0 \dots i_n})$$

define

$$d : \mathcal{E}^n(\mathcal{U}, \mathbb{F}) \longrightarrow \mathcal{E}^{n+1}(\mathcal{U}, \mathbb{F})$$

by

$$a = (a_{i_0, \dots, i_n})_{i_0 < \dots < i_n} \in \mathcal{E}^n(\mathcal{U}, \mathbb{F})(V), \quad V \subset X \text{ open}$$

$$\leadsto da := (da)_{i_0, \dots, i_{n+1}}_{i_0 < \dots < i_{n+1}}$$

with

$$(da)_{i_0, \dots, i_{n+1}} := \sum_{k=0}^{n+1} (-1)^k a_{i_0, \dots, \overset{\uparrow}{\underset{\text{omit}}{i_k}}, \dots, i_{n+1}} | \mathcal{U}_{i_0, \dots, i_{n+1}} \cap V$$

$$\in \mathbb{F}(\mathcal{U}_{i_0, \dots, i_{n+1}} \cap V)$$

Q. 10a. (X, A) , F , \mathcal{U} as above

$$\Rightarrow 0 = \text{dod} : \begin{aligned} \mathcal{E}^n(\mathcal{U}, F) &\longrightarrow \mathcal{E}^{n+2}(\mathcal{U}, F) \\ \mathcal{C}^n(\mathcal{U}, F) &\longrightarrow \mathcal{C}^{n+2}(\mathcal{U}, F) \end{aligned}$$

pf.

$$a = (a_{i_0, \dots, i_{n+2}})$$

$$(d da)_{i_0, \dots, i_{n+2}} = \sum_{k=0}^{n+2} (-1)^k (da)_{i_0, \dots, \overset{\uparrow}{i_k}, \dots, i_{n+2}} | \mathcal{U}_{i_0, \dots, i_{n+2}}$$

$$= \sum_{k=0}^{n+2} (-1)^k \sum_{l=0}^{k-1} (-1)^l a_{i_0, \dots, \overset{\uparrow}{i_l}, \dots, \overset{\uparrow}{i_k}, \dots, i_{n+2}} | \mathcal{U}_{i_0, \dots, i_{n+2}}$$

$$+ \sum_{k=0}^{n+2} (-1)^k \sum_{l=k+1}^{n+2} (-1)^{l-1} a_{i_0, \dots, \overset{\uparrow}{i_k}, \dots, \overset{\uparrow}{i_l}, \dots, i_{n+2}} | \mathcal{U}_{i_0, \dots, i_{n+2}}$$

$$= \sum_{0 \leq r < s \leq n+2} \left((-1)^{s+r} + (-1)^{r+s-1} \right) a_{i_0, \dots, \overset{\uparrow}{i_r}, \dots, \overset{\uparrow}{i_s}, \dots, i_{n+2}} | \mathcal{U}_{i_0, \dots, i_{n+2}}$$

$$= 0 \quad \square$$

③ Def (X, A) ringed space, $F \in \mathcal{S}\mathcal{F}(X, A)$

$\mathcal{U} = (U_i)_{i \in I}$ open cover, $(I, <) \text{ order}$

the (sheaf) Čech complex of (F, \mathcal{U}) is

$$\mathcal{C}^\bullet(\mathcal{U}, F) = \mathcal{C}^0(\mathcal{U}, F) \xrightarrow{d} \mathcal{C}^1(\mathcal{U}, F) \xrightarrow{d} \mathcal{C}^2(\mathcal{U}, F) \xrightarrow{d} \dots$$

the (global) Čech complex is

$$C^\bullet(\mathcal{U}, F) = C^0(\mathcal{U}, F) \xrightarrow{d} C^1(\mathcal{U}, F) \xrightarrow{d} C^2(\mathcal{U}, F) \xrightarrow{d} \dots$$

$$= \Gamma(X, \mathcal{C}(\mathcal{U}, F))$$

the n-th Čech cohomology is

$$\check{H}^n(\mathcal{U}, F) = H^n(C^\bullet(\mathcal{U}, F))$$

$\in (A(X)\text{-mod})$

The Čech complex comes with a natural augmentation

$$\varepsilon : F \rightarrow \mathcal{C}^0(\mathcal{U}, F) \text{ given by } F(U) \rightarrow \mathcal{C}^0(\mathcal{U}, F)(U) = \prod_{i \in I} F(U_i) \\ \simeq \mapsto (s|_{U_i})_{i \in I}$$

④ Ex:

$$i) H^0(\mathcal{U}, F) = \ker \left(\prod_{i \in I} F(U_i) \rightarrow \prod_{i < j} F(U_i \cap U_j) \right)$$

$$(a_i) \mapsto (a_i|_{U_i \cap U_j} - a_j|_{U_i \cap U_j})$$

$$= F(X)$$

F sheaf

(ii) $\mathcal{U} = \{U, V, W\}$

$\begin{matrix} \parallel & \parallel & \parallel \\ U_0 & U_1 & U_2 \end{matrix}$

$$C(\mathcal{U}, F) = F(U) \times F(V) \times F(W) \rightarrow F(U \cap V) \times F(U \cap W) \times F(V \cap W)$$

$$\rightarrow F(U \cap V \cap W) \rightarrow 0$$

$$(a, b, c) \mapsto (b|_{U \cap V} - a|_{U \cap V}, c|_{U \cap W} - a|_{U \cap W}, c|_{V \cap W} - b|_{V \cap W})$$

$$(d, \beta, \alpha)$$

$$\mapsto \alpha|_{U \cap V \cap W} - \beta|_{U \cap V \cap W} + d|_{U \cap V \cap W}$$

Composition:

$$(a, b, c) \mapsto (c|_{U \cap W} - b|_{U \cap W}) - (c|_{U \cap W} - a|_{U \cap W}) + (b|_{U \cap W} - a|_{U \cap W})$$

$$= 0$$

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construct homotopy between $\varepsilon \circ d$ and id_C^0

$s^0 = 0$

define $s^m : C^m(\mathcal{U}, F) \rightarrow C^{m-1}(\mathcal{U}, F)$

\parallel

$\prod_{i_0 < \dots < i_m} F(\mathcal{U}_{i_0 \dots i_m}) \quad \parallel \quad \prod_{i_0 < \dots < i_{m-1}} F(\mathcal{U}_{i_0 \dots i_{m-1}})$

by

$$s^m(a)_{i_0 \dots i_{m-1}} = \begin{cases} 0, & \text{if } x \in \{i_0, \dots, i_{m-1}\} \\ (-1)^{q-1} a_{i_0, \dots, i_{q-1}, x, i_{q+1}, \dots, i_{m-1}} & \text{if } i_{q-1} < x < i_q \\ & (i_{-1} = -\infty, i_n = \infty) \end{cases}$$

$a = (a_{i_0 \dots i_{m-1}})$

well-defined since

$$\begin{aligned} \mathcal{U}_{i_0, \dots, i_{q-1}, x, i_{q+1}, \dots, i_{m-1}} &= \mathcal{U}_{i_0, \dots, i_{m-1}} \cap \mathcal{U}_x \\ &= \mathcal{U}_{i_0, \dots, i_{m-1}} \end{aligned}$$

claim:

$$\varepsilon \circ d - \text{id}_C^0 = s \circ d + d \circ s$$

Indeed:
 $\alpha = (a_{i_0}) \in C^0(\mathcal{M}, \mathbb{F})$

$$\Rightarrow ((\mathcal{E} \circ d)^0 - id_{C^0}) (a)_{i_0} = a_{x|u_{i_0}} - a_{i_0}$$

$$(s d + d s)^0 (a)_{i_0} = (s^1 d a)_{i_0} = s^1 \left((a_{x|u_{i_0}} - a_{i_0})_{i_0 < i_1} \right)_{i_0}$$

$$= \begin{cases} 0 & i_0 = \mathcal{K} \\ (-1)^0 a_{x|u_{i_0}} - a_{i_0} & i_0 < \mathcal{K} \\ (-1)^{-1} a_{i_0} - a_{x|u_{i_0}} & i_0 > \mathcal{K} \end{cases}$$

$$= ((\mathcal{E} \circ d)^0 - id_{C^0}) (a)_{i_0}$$

Assume $n \gg 1 \Rightarrow |\mathcal{E} \circ d|^n = 0$

$$(sd + ds)^n (a)_{i_0, \dots, i_n} = (s^{n+1}d^n + d^{n+1}s^n) (a)_{i_0, \dots, i_n}$$

$$a = (a_{i_0, \dots, i_n})$$

1. case $K \in \{i_0, \dots, i_n\}$

say $K = i_x$

$$\Rightarrow (s^{n+1}d^n (a))_{i_0, \dots, i_n} = 0$$

$$(d^{n+1}s^n (a))_{i_0, \dots, i_n} = \sum_{l=0}^n (-1)^l \underbrace{s^n (a)_{i_0, \dots, i_{l-1}, i_{l+1}, \dots, i_n}}_{=0 \text{ if } l \neq x} \Big| a_{i_0, \dots, i_n}$$

$$= (-1)^x (-1)^{x-1} a_{i_0, \dots, i_{x-1}, i_{x+1}, \dots, i_n}$$

$$= - a_{i_0, \dots, i_n}$$

$$= (\epsilon^{\partial} - i^{\partial}_c) (a)_{i_0, \dots, i_n}$$

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2. case: $\lambda \notin \{i_0, \dots, i_n\}$

say $i_{q-1} < \lambda < i_q$

$q \in \{0, \dots, n+1\}$

$$\Rightarrow (S^{q+1} d^n(a))_{i_0, \dots, i_n} = (-1)^{q-1} (d^n(a))_{i_0, \dots, i_{q-1}, \lambda, i_q, \dots, i_n}$$

$$(d^{q-1} S^n(a))_{i_0, \dots, i_n} = \sum_{l=0}^n (-1)^l S^n(a)_{i_0, \dots, i_{l-1}, \lambda, i_{l+1}, \dots, i_n}$$

$$= \sum_{l=0}^{q-1} (-1)^l (-1)^{q-2} a_{i_0, \dots, i_{l-1}, i_{q-1}, \lambda, i_{q+1}, \dots, i_n} | \mathcal{U}_{i_0, \dots, i_n}$$

$$+ \sum_{l=q}^n (-1)^l (-1)^{q-1} a_{i_0, \dots, i_l, i_{q-1}, \lambda, i_{q+1}, \dots, i_n} | \mathcal{U}_{i_0, \dots, i_n}$$

$$\Rightarrow (S^{q+1} d^n + d^{q-1} S^n) | \mathcal{U}_{i_0, \dots, i_n} = (-1)^{q-1} (-1)^q a_{i_0, \dots, i_n} \\ = (\epsilon_0 d - id_c)^n | \mathcal{U}_{i_0, \dots, i_n}$$

Thus $\epsilon_0 d \sim id_c$. \square

⑥ Prop. $F \in \mathcal{S}\mathcal{G}(X, A)$, $\mathcal{U} = (U_i)_{i \in I}$

$F \xrightarrow{\varepsilon} \mathcal{E}^\bullet(\mathcal{U}, F)$ is a resolution
 (see ③) in $\mathcal{S}\mathcal{G}(X, A)$

Pf. let $x \in X$. It suffices to check:

$F_x \rightarrow \mathcal{E}^\bullet(\mathcal{U}, F|_x)$ is a resolution
 in A_x -mod.

\mathcal{U} cover \Rightarrow

$\exists K \subset I$ s.t. $x \in U_K$

let $x \in V \subset U_K$
 open

\rightarrow it suffices to check

$F(V) \rightarrow \mathcal{E}^\bullet(\mathcal{U}, F|_V) = \mathcal{E}^\bullet(\mathcal{U} \cap V, F)$
 is a resolution
 ($\mathcal{U} \cap V = (U_i \cap V)_{i \in I}$)

This follows from (5) \square

(7) For $F \in \mathcal{S}(X, A)$, $\mathcal{V} = (\mathcal{V}_i)$
 $\forall n \geq 0 \exists$ map

$$H^n(\mathcal{V}, F) \rightarrow H^n(X, F)$$

functorial in F , isom for $n=0$

pf:

$$\begin{array}{ccc}
 F & \rightarrow & \mathcal{C}^*(\mathcal{V}, F) \text{ (functional)} \\
 \parallel & & \downarrow \exists \text{ unique up to homotopy (see § 2, (9))} \\
 F & \rightarrow & \mathcal{Z}^*(F) \\
 & & \text{(functional inj res)}
 \end{array}$$

□

(8) Recall X top space

a base of the topology of X is a collection

\mathcal{B} of open subset of X s.t.

$\forall \mathcal{U} \subset X$ open we can write $\mathcal{U} = \bigcup_{i \in I} V_i$
 with $V_i \in \mathcal{B}$

(9) Let: (X, A) ringed space

\mathcal{B} a base of the topology of X s.t.

- (i) $V \in \mathcal{B} \Rightarrow V$ quasi-compact
- (ii) $\emptyset \in \mathcal{B}$
- (iii) $V, W \in \mathcal{B} \Rightarrow V \cap W \in \mathcal{B}$

Let $\mathcal{D} \subset \mathcal{S}_A(X, A)$ be the full subcategory consisting of $F \in \mathcal{S}_A(X, A)$ s.t.

$\forall U \in \mathcal{B}$ and all $\mathcal{U} = (U_i)_{i=1, \dots, r}$ ^{finite open cover of U} with $U_i \in \mathcal{B}$ ^($U = \bigcup U_i$)

we have $H^m(U, F|_U) = 0 \quad \forall m \geq 1$

Then $H^m(U, F|_U) = 0 \quad \forall F \in \mathcal{D}$ and all $U \in \mathcal{B}$

Pf. We first prove the following properties of \mathcal{D} :

(1) $F \in \mathcal{S}(X, A)$ flasque $\Rightarrow F \in \mathcal{D}$

(2) $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ s.e.s in $\mathcal{S}(X, A)$, $F' \in \mathcal{D}$

$\Rightarrow 0 \rightarrow T(u, F') \rightarrow T(u, F) \rightarrow T(u, F'') \rightarrow 0$ exact $\forall u \in \mathcal{B}$

(3) $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ s.e.s in $\mathcal{S}(X, A)$

and $F', F \in \mathcal{D}$

$\Rightarrow F'' \in \mathcal{D}$

ad (1): F flasque, $u \in \mathcal{B}$, $u = \bigcup_{i=1}^r u_i$, $u_i \in \mathcal{B}$
 $\mathcal{U} = (u_i)_{i=1, \dots, r}$

$\Rightarrow j_*(F|_{u_{i_1, \dots, i_n}})$ is flasque on u

$\Rightarrow \mathcal{E}(\mathcal{U}, F) = \prod_{1 \leq i_1 < \dots < i_n \leq r} j_*(F|_{u_{i_1, \dots, i_n}})$ flasque

\Rightarrow $F|_u \rightarrow \mathcal{E}(\mathcal{U}, F|_u)$ is a flasque resolution

⑥

\Rightarrow
§4, (17)

$$H^m(U, F|_U) = H^m(\pi(U, \mathcal{C}(\mathcal{U}, F|_U)))$$

$$\cong H^m(\mathcal{U}, F|_{\mathcal{U}})$$

$F|_U$ flasque
(§4, (10))

$\Rightarrow F \in \mathcal{D}$

ad(2): $0 \rightarrow F' \xrightarrow{a} F \xrightarrow{b} F'' \rightarrow 0$ s.t. $s, F \in \mathcal{D}$

to show:

$$\pi(U, F) \xrightarrow{b|_U} \pi(U, F'')$$

surj $\forall U \in \mathcal{B}$

indeed: $s \in \pi(U, F)$

\exists base of top, U quasi-compact

$\Rightarrow \exists$ open cover $\mathcal{U} = (U_i)_{i=1, \dots, r}$ of U
 $U_i \in \mathcal{B}$

and $t_i \in F(U_i)$ s.t.

$$b(t_i) = s|_{U_i}$$

$\Rightarrow \exists u_{ij} \in F(U_{ij})$ s.t. $a(u_{ij}) = t_j|_{U_{ij}} - t_i|_{U_{ij}}$

$$\mu := (\mu_{ij}) \in \prod_{1 \leq i < j \leq r} F(u_{ij})$$

$$(d\mu)_{ijk} = \mu_{jk}|_{u_{ijk}} - \mu_{ik}|_{u_{ijk}} + \mu_{ij}|_{u_{ijk}} = 0$$

(since $\alpha(d\mu)_{ijk}|_{u_{ijk}} = 0$ and $\alpha(u_{ij})$)

$$\Rightarrow \mu \text{ define elt in } H^1(\mathcal{U}, F) = 0$$

\uparrow
 $F \in \mathcal{D}$

$$\Rightarrow \exists \mu_i \in F(u_i) \quad i=1, \dots, r \text{ s.t.}$$

$$\mu_{ij} = \mu_j - \mu_i|_{u_{ij}}$$

$$\text{set } \tau_i := t_i - \alpha(\mu_i) \in F(u_i)$$

$$\begin{aligned} \Rightarrow \tau_i|_{u_{ij}} &= t_i|_{u_{ij}} - \alpha(\mu_i)|_{u_{ij}} = t_j|_{u_{ij}} - \alpha(\mu_i)|_{u_{ij}} - \alpha(\mu_j)|_{u_{ij}} \\ &= t_j|_{u_{ij}} - \alpha(\mu_j)|_{u_{ij}} = \tau_j|_{u_{ij}} \end{aligned}$$

$$\Rightarrow \exists \tau \in F(u) \text{ with } \tau|_{u_i} = t_i - \alpha(\mu_i)$$

$$\Rightarrow b(\tau)|_{u_i} = b(t_i)|_{u_i} = s \cdot \mu_i \quad \forall i \Leftrightarrow b(\tau) = s = \sum \mu_i \Leftrightarrow b(u) \text{ sur}$$

(ad 3) $0 \rightarrow F' \rightarrow F \rightarrow F'' \rightarrow 0$ s. o. s

$F', F \in \mathcal{D}$

to show: $F'' \in \mathcal{D}$

pf: let $U \in \mathcal{B}$ and $\mathcal{U} = (U_i)_{i=1, \dots, r}$
open cover of U
with $U_i \in \mathcal{B}$

By assumption $U_{i_0, \dots, i_n} \in \mathcal{B} \quad \forall i_0, \dots, i_n$

(2) \Rightarrow

$0 \rightarrow C(\mathcal{U}, F') \rightarrow C(\mathcal{U}, F) \rightarrow C(\mathcal{U}, F'') \rightarrow 0$
is exact.

\Rightarrow
l.o.s
(§1, (5)) $\dots \rightarrow H^n(\mathcal{U}, F') \rightarrow H^n(\mathcal{U}, F) \rightarrow H^{n+1}(\mathcal{U}, F'') \rightarrow \dots$
 $\quad \quad \quad \parallel \leftarrow F' \in \mathcal{D}$
 $\quad \quad \quad 0$

$\Rightarrow H^n(\mathcal{U}, F'') = 0 \quad \forall n \geq 1$
 $\Rightarrow F'' \in \mathcal{D}$

We proved (1) - (3)

now to show $H^i(U, F) = 0 \quad \forall n \geq 1, U \in \mathcal{B}$
 $F \in \mathcal{D}$

$F \in \mathcal{D}$

$$0 \rightarrow F \rightarrow I \rightarrow C \rightarrow 0$$

\uparrow
 $\text{inj} \Rightarrow \text{flasque} \quad \Rightarrow C \in \mathcal{D} \quad (3)$
 $\Rightarrow I \in \mathcal{D} \quad (1)$

$$\Rightarrow \pi(U, I) \rightarrow \pi(U, C) \rightarrow H^1(U, F) \rightarrow H^1(F, \mathbb{Z})$$

$\text{surj} \quad \parallel$
 $\hookrightarrow (2) \quad 0$

$$\Rightarrow H^1(U, F) = 0 \quad \forall U, F$$

Assume we have shown $H^i(U, F) = 0 \quad \forall 1 \leq i \leq n-1$
 $\forall U, F$
 $n \geq 2$

$$\Rightarrow \dots \rightarrow H^{n-1}(U, C) \rightarrow H^n(U, F) \rightarrow H^n(U, I) \rightarrow \dots$$

$\parallel \text{Ind} \quad \parallel \text{inj}$
 $0 \in \mathcal{D} \quad \Rightarrow H^n(U, F) = 0 \quad \square$

