

§ 5 Grothendieck Vanishing Theorem on noetherian topological spaces

① Recall. Let (A, \leq) be a partially ordered set.

We say A is directed if

$$\forall \alpha, \beta \in A \quad \exists \gamma \in A \quad \alpha, \beta \leq \gamma$$

• let \mathcal{E} be a category

a directed system on a directed system (A, \leq)

is a collection $(X_\alpha)_{\alpha \in A}$

with maps $X_\alpha \xrightarrow{\bar{\iota}_{\alpha\beta}} X_\beta \quad \forall \alpha \leq \beta$

$$s.t. \text{ i)} \quad \bar{\iota}_{\alpha\alpha} = \text{id}_{X_\alpha}$$

$$\text{ii)} \quad \bar{\iota}_{\beta\beta} \circ \bar{\iota}_{\alpha\beta} = \bar{\iota}_{\alpha\beta} \quad \begin{matrix} \text{one pt set} \\ \downarrow \end{matrix}$$

↑ equiv: view (A, \leq) as a cat with $\{ \uparrow \}, \alpha \leq \beta \}$
 $\text{obj}(A) = A$ and $\text{Hom}(d, \beta) = \begin{cases} \{ \uparrow \}, & d \leq \beta \\ \emptyset, & \text{else} \end{cases}$

then $(X_\alpha)_{\alpha \in A} \hookrightarrow \text{functor } X : A \rightarrow \mathcal{E}$

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• let $(X_\alpha)_{\alpha \in A}$ be a direct system

the direct limit (or colimit)

of $(X_\alpha)_{\alpha \in A}$ is (if it exists)

an object $Y \in \mathcal{C}$ with maps

$$\sigma_\alpha : X_\alpha \rightarrow Y$$

s.t.

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\sigma_\alpha} & Y \\ \downarrow \tau_{\alpha\beta} & \nearrow \gamma & \\ X_\beta & \xrightarrow{\sigma_\beta} & \end{array} \quad \forall \alpha \leq \beta$$

satisfying the following universal property :

$$\forall Z \in \mathcal{C} \text{ with map } \begin{array}{ccc} X_\alpha & \xrightarrow{\varphi_\alpha} & Z \\ \downarrow \tau_{\alpha\beta} & \nearrow \varphi_\beta & \\ X_\beta & \xrightarrow{\varphi_\beta} & \end{array} \quad \forall \alpha \leq \beta$$

$$\exists! \quad \varrho : Y \rightarrow Z \text{ s.t. } \varrho_\alpha = \varrho \circ \sigma_\alpha$$

If the direct limit exists it is unique up to unique isomorphism. We denote it by $\varinjlim_{\alpha \in A} X_\alpha$
(or $\operatorname{colim}_A X_\alpha$)

⑦ Lend:

(1) R ring | $(M_\alpha, \tau)_\alpha \in A$ directed sys of
 (left) R-modules

$$\Rightarrow \varinjlim_{\alpha \in A} M_\alpha = \frac{\{m \in M_\alpha \text{ for some } \alpha \in A\}}{\sim} \in (R\text{-Mod})$$

where $m \in M_\alpha \sim m \in M_\beta : \Leftrightarrow \alpha \leq \beta \text{ and } \tau_{\alpha \beta}^{(m)} = m$

R-mod str via

$$m_\alpha + x m_\beta := \tau_{\alpha \beta}(m_\alpha) + x \tau_{\beta \alpha}(m_\beta) \quad \text{for } x \in R$$

(2) (X, \mathcal{O}) ringed space, $(\mathcal{F}_\alpha)_{\alpha \in A}$ directed sys in $\mathcal{S}\mathcal{C}(X, \mathcal{O})$

Then $\varinjlim_{\alpha \in A} \mathcal{F}_\alpha \in \mathcal{S}\mathcal{C}(X, \mathcal{O})$ exists

and it has the following properties

(i) $\lim_{\substack{\rightarrow \\ \mathcal{I}}} F_x$ is the sheaf on X associated to

$$X \underset{\text{open}}{\supset} U \mapsto \lim_{\substack{\rightarrow \\ \mathcal{I}}} (F_x(U))$$

↑
colimit in $(\text{Ab}-\text{Mod})$

| ii) if $(F_\alpha), (F'_\alpha), (F''_\alpha)$ one direct sys in $\mathcal{S}(X, 0)$

and

$$0 \rightarrow F_\alpha^1 \rightarrow F_\alpha \rightarrow F_\alpha'' \rightarrow 0 \text{ exact } \forall \alpha \in A$$

$$\Rightarrow 0 \rightarrow \lim_{\substack{\rightarrow \\ \alpha}} F_\alpha^1 \rightarrow \lim_{\substack{\rightarrow \\ \mathcal{I}}} F_\alpha \rightarrow \lim_{\substack{\rightarrow \\ \mathcal{I}}} F_\alpha'' \rightarrow 0 \text{ is exact.}$$

If: direct to check \triangleright

(3) Recall

X top space

* X is called noetherian if

any descending chain of closed
subsets in X

$$Z_1 \supset Z_2 \supset Z_3 \supset \dots \supset Z_n \supset Z_{n+1} \supset \dots$$

becomes stationary

Examples: Noetherian schemes / Non-examples: manifolds

(4) Exa: (X, \mathcal{O}) noetherian top ringed space

$(\mathcal{F}_\alpha)_{\alpha \in A}$ direct system in $\mathcal{Sh}(X, \mathcal{A})$

Then: $T^*(U, \varinjlim_{\alpha \in A} \mathcal{F}_\alpha) = \varinjlim_{\alpha \in A} T^*(U, \mathcal{F}_\alpha)$

$\forall U \subset X$ open

Pf: We show

$$U \mapsto \varinjlim_{\alpha \in A} T(U, \mathbb{F}_\alpha) \quad \text{is a sheaf.}$$

Note X noeth top space $\Rightarrow \forall U \subset X :$

U is quasi-compact
with induced top.

1. Sheaf axiom: $U \subset X$ open, set $\varinjlim_{\alpha \in A} T(U, \mathbb{F}_\alpha)$

$U = \bigcup_i U_i$ open cover with

$$s|_{U_i} = 0 \quad \forall i$$

$\textcircled{2} \Rightarrow \exists x_0 \in A$ and $\tilde{s} \in T(U, \mathbb{F}_{x_0})$ s.t.

$$\tilde{s} \mapsto s$$

U q. comp $\Rightarrow \exists U = \bigcup_{q=1}^n U_q$ fin open subcover

$$s|_{U_q} = 0 \Rightarrow \exists x_q : \tau(\tilde{s})|_{U_q} = 0 \quad q = 1, \dots, n$$

A directed $\Rightarrow \exists \beta \geq x_1, x_2, \dots, x_n$

$$\Rightarrow \tau_{x_0 \beta}(\tilde{s})|_{U_q} = 0 \quad \forall q \Rightarrow \tau_{x_0 \beta}(\tilde{s}) = 0$$

$$\Rightarrow s = 0 \text{ in } \varinjlim_{\alpha \in A} T(U, \mathbb{F}_\alpha)$$

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2nd Shaf axiom:

$$\mathcal{U} = \bigvee_i \mathcal{U}_i, \quad s_i \in \lim_{\rightarrow} T(\mathcal{U}_i, \mathbb{F}_{\alpha})$$

$$\text{with } s_i|_{\mathcal{U}_{ij}} = s_j|_{\mathcal{U}_{ij}}. \quad \mathcal{U}_{ij} := \mathcal{U}_i \cap \mathcal{U}_j$$

\mathcal{U} quasi compact $\Rightarrow \mathcal{U} = \bigcup_{g=1}^r \mathcal{U}_g$ fin open subcover

$$\exists \quad \tilde{s}_g \in T(\mathcal{U}_g, \mathbb{F}_{\alpha_g}) \text{ s.t. } \tilde{s}_g \mapsto s_g$$

$$s_g|_{\mathcal{U}_{g,l}} = s_l|_{\mathcal{U}_{g,l}} \Rightarrow \exists \beta \ni \alpha_1, \dots, \alpha_g \text{ s.t.}$$

$$T_{\alpha_g \beta}(\tilde{s}_g)|_{\mathcal{U}_{g,l}} = T_{\alpha_g \beta}(\tilde{s}_l)|_{\mathcal{U}_{g,l}} \quad \forall g, l$$

$$\Rightarrow \exists \quad \tilde{s} \in T(\mathcal{U}, \mathbb{F}_{\beta}) \quad \text{with}$$

$$\tilde{s}|_{\mathcal{U}_g} = \tilde{s}_g$$

$$s = \text{im } \tilde{s} \text{ in } \lim_{\rightarrow} T(\mathcal{U}, \mathbb{F}_{\alpha})$$

$$\Rightarrow s|_{\mathcal{U}_g} = s_g \quad \forall g = 1, \dots, n$$

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check for s_i :

$$U_i = \bigcup_{g=1}^{\infty} U_{gi} \cap U_i$$

$$s_i|_{U_{gi}} = s_g|_{U_{gi}} = s|_{U_{gi}} \quad \forall g \quad \forall i$$

$\stackrel{?}{=} \quad s_i = s|_{U_i} \quad \forall i$

□

⑤ (X, \mathcal{O}) metrisable top ringed space

$(\mathbb{F}_d)_{d \in A}$ direct system in $\text{Sh}(X, \mathcal{O})$

Assume \mathbb{F}_d is flasque $\forall d$

$\Rightarrow \varinjlim_{d \in A} \mathbb{F}_d$ flasque

Pf: $V \subset U \subset X$ open

flasque $\Rightarrow \forall d : \mathbb{F}_d(U) \rightarrow \mathbb{F}_d(V)$ surj.

$\Rightarrow \varinjlim_{d \in A} \mathbb{F}_d(U) \rightarrow \varinjlim_{d \in A} \mathbb{F}_d(V)$ surj.

$(\varinjlim_{d \in A} \mathbb{F}_d(U), \varinjlim_{d \in A} \mathbb{F}_d(V))$ surj.

$(T/U, \varinjlim_{d \in A} \mathbb{F}_d)$

□

(6) Prop: (X, \mathcal{O}) noeth top ringed space
 $(\mathbb{F}_d)_{d \in A}$ direct system in $\mathcal{SH}(X, \mathcal{O})$

Then

$$\varinjlim_{a \in A} H^i(X, \mathbb{F}_d) = H^i\left(X, \varinjlim_{a \in A} \mathbb{F}_d\right), \text{ Higo}$$

Note if (\mathbb{F}_d) is a direct sys in \mathcal{E}

and $T: \mathcal{E} \rightarrow D$ is a functor

$\Rightarrow (T(\mathbb{F}_d))$ is a direct system in D

Pf: set $\mathcal{SH}(X, \mathcal{O})^{(A, \leq)} =$ category of direct systems indexed by A
 $(= \text{functors } A \rightarrow \mathcal{SH}(X, \mathcal{O}))$

This is an abelian category (check!)

Set $s^i := \lim_{\substack{\rightarrow \\ x \in A}} H^i(x, -)$

$$T^i := H^i(x, \lim_{\substack{\rightarrow \\ x \in A}} (-))$$

$$\Rightarrow s^i, T^i : \mathcal{SH}(X, \mathcal{O})^{(A, \leq)} \rightarrow (\mathcal{O}(X)\text{-Mod})$$

functions

since $\lim_{\substack{\rightarrow \\ x \in A}} :$

$\mathcal{SH}(X, \mathcal{O})^{(A, \leq)}$	$\rightarrow \mathcal{SH}(X, \mathcal{O})$
$(\mathcal{O}(X)\text{-Mod})^{(A, \leq)}$	$\rightarrow (\mathcal{O}(X)\text{-Mod})$
is exact	
(see ②)	

we obtain s -functors

$$(s^i)_{i \geq 0}, (T^i)_{i \geq 0}$$

with $s^0 = \lim_{\substack{\rightarrow \\ x \in A}} T(x, -) \quad \textcircled{4} \quad T(x, \lim_{\substack{\rightarrow \\ x \in A}} (-)) = T^0$

It suffices to show both are effaceable

then by § 3, ⑥ both are universal and hence
are canonically isomorphic.

for $F \in \mathcal{S}(\chi, \theta)$

set $G(F)(U) := \prod_{x \in U} F_x$, U $\subset X$
open
shell

$\Rightarrow G(F) \in \mathcal{S}(\chi, \theta)$

* $G(F)$ is plasque

* $F \rightarrow G(F)$ nat map is injective

S^i effable: $(F_\alpha)_{\alpha \in A} \rightarrow (G(F_\alpha))_{\alpha \in A}$
monic in $\mathcal{S}(\chi, \theta)^{A, \leq 1}$

and

$S^i((G(F_\alpha))_X) = \varinjlim_{\alpha \in A} H^i(X, G(F_\alpha)) \stackrel{\text{def}}{=} 0$ $H^i, 1$
 $\uparrow (64, 10)$

T^i effable:

$T^i((G(F_\alpha))_{\alpha \in A}) = H^i(X, \varinjlim_{\alpha \in A} G(F_\alpha)) = 0$
plasque by (5) \square

(8) Dimension (Grothendieck) X with top space

\mathcal{F} sheaf of abelian groups on X , i.e. $\mathcal{F} \in \mathbf{Sh}(X, \mathbb{Z})$

$$\Rightarrow H^j(X, \mathcal{F}) = 0 \quad \forall j > \dim X$$

(Recall: $\dim X = \sup \left\{ \begin{array}{l} |I| \neq \emptyset \subseteq \mathbb{Z}_1 \cup \dots \cup \mathbb{Z}_r \\ \mathbb{Z}_i \subset X \text{ closed} \end{array} \right\}$)

Pf. (By atomization)

Note statement is empty if $\dim X = \infty$

\rightarrow wlog $\dim X = n < \infty$

Recall $\mathcal{U} \xrightarrow[\text{open}]{} X \xrightarrow[\text{closed.}]{} \mathcal{Z} := X \setminus \mathcal{U}$, $\mathcal{F} \in \mathbf{Sh}(X, \mathbb{Z})$

\Rightarrow exact seq

$$0 \rightarrow j_! j^{-1} \mathcal{F} \rightarrow \mathcal{F} \rightarrow i_* (i^{-1} \mathcal{F}) \rightarrow 0 \quad (\star)$$

see § 4, ③

(check on stalks)

1. Step: wlog X irrecl.

since: X noeth \Rightarrow it has fin many
irred cptls

let $Z \subset X$ irred cptl

$$U = X \setminus Z \hookrightarrow X$$

note $U \hookrightarrow \bar{U} \xrightarrow{i} X$, \bar{U} has less irred
cptls than X

§4, (14) \Rightarrow

$$H^a(X, i_*(i^{-1}\mathbb{F})) = H^a(Z, i^{-1}\mathbb{F})$$

$$H^a(X, j_!(j^{-1}\mathbb{F})) = H^a(X, i_* i_! (i^{-1}\mathbb{F})) = H^a(\bar{U}, i_! (j^{-1}\mathbb{F}))$$

by induction on # irred cptl these vanish for
 $a > \dim X \geq \dim Z, \dim \bar{U}$

$$H^a(X, \mathbb{F}) = 0 \quad \forall a > \dim X$$

\Rightarrow
l.e.s
for i^*

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Now we do find on the dim of X

2. Step $\dim X = 0 \Rightarrow$ Then OK

since: X is not + $\dim X = 0$

\Rightarrow the only opens are $\{\emptyset, X\}$

(else $\exists U \neq \emptyset \neq X$ closed \Downarrow
 $\dim X = 0$)

$\Rightarrow T(X, -) : \mathcal{S}\mathcal{H}(X) \rightarrow (\text{ab gps})$

$\mathbb{F} \mapsto T(X, \mathbb{F})$

is an equivalence
of categories

Hence is exact $\Rightarrow H^q(X, \mathbb{F}) = 0$
 $\forall q > 1$

3. Step: Assume $\dim X = n \geq 1$ and X is not
Then it suff to prove and vanishing true for $\dim < n$

($\exists \mathbb{F} \quad H^q(X, j_! \mathbb{Z}_U) = 0 \quad \forall q > n$ and all $U \subset X$
open)

where $\mathbb{Z}_U = \text{constant sheaf def by } \mathbb{Z} \text{ on } U$.

instead:

Assume no Riesz (\star)

let $F \in SR(X)$

set $B := \{ (U, s) \mid U \subset X \text{ open}, s \in F(U) \}$

$A := \{ \alpha \subset B \mid |\alpha| < \infty \}$

$\Rightarrow A$ directed set $[\alpha \leq \alpha' \Leftrightarrow \alpha \subset \alpha']$

Recall $s \in F(U) \hookrightarrow j_!(\mathbb{Z}_U) \xrightarrow{s} F$
 $j: U \subset X$ §4, (3), vii

for $\alpha = ((U_1, s_1), \dots, (U_r, s_r))$

set $F_\alpha := \text{Im} \left(\bigoplus_{l=1}^r j_{\alpha_l}^! (\mathbb{Z}_{U_l}) \xrightarrow{\{s_i\}} F \right)$

$\Rightarrow (F_\alpha)_{\alpha \in A}$ direct system

($\alpha \leq \alpha' \Rightarrow F_\alpha \subset F_{\alpha'} \subset F$)

and $F = \varinjlim_{\alpha} F_\alpha$

\Rightarrow it suff to show

$$(6) \quad H^a(X, \mathcal{F}_\alpha) = 0 \quad \forall a > n$$

\Rightarrow it suff to show the vanishing

for fin gen sheaf, i.e.

sheaves F with a surj

$$\bigoplus_{l=1}^r j_{l!}(\mathbb{Z}_{u_l}) \rightarrow F$$

$$\text{set } F_1 = \text{im } (j_{n!} \mathbb{Z}_{u_n} \rightarrow F)$$

$$\Rightarrow 0 \rightarrow F_1 \rightarrow F \rightarrow G \rightarrow 0$$

$\downarrow \quad \uparrow$

has 1 generator has $r-1$ generators

\Rightarrow it suff to show
Ind/generators

$$+ l.e.s. \quad H^a(X, F) = 0 \quad \forall a > n$$

F F with $j_!(\mathbb{Z}_n) \rightarrow F$ some $u \in X$
open

\Rightarrow Have

$$0 \rightarrow K \xrightarrow{\quad} j_!(\mathbb{Z}_U) \rightarrow \mathbb{F} \rightarrow 0 \quad (13*)$$

\downarrow
Kernel

and we suppose we have shown

$$(12*) \quad H^\alpha(X, j_!\mathbb{Z}_U) = 0 \quad \forall \alpha > n, \forall U$$

$$\underline{\text{if } K = 0 \Rightarrow H^q(X, \mathbb{F}) = 0 \quad \forall q > n.}$$

$$\underline{\text{if } K \neq 0 \quad \text{set } N := \min \{ m \in \mathbb{Z}_{\geq 1} \mid \exists x \in U \text{ s.t. } }$$

$m \in K_x \subset \mathbb{Z}$

\uparrow
stall

$\Rightarrow \exists V \subset U \text{ open s.t.}$

definition
of stall

$$K|_V = N \cdot \mathbb{Z}_V \cong \mathbb{Z}_V$$

$$\Rightarrow 0 \rightarrow j_!(\mathbb{Z}_V) \rightarrow K \rightarrow \frac{K}{j_!(\mathbb{Z}_V)} \rightarrow 0$$

\uparrow
this has

support in $U \setminus V = Y$

$\dim Y < n$ (since U is closed)

By induction $H^\alpha(X, \frac{K}{j_!(\mathbb{Z}_V)}) = H^\alpha(Y, i_{Y*}(\frac{K}{j_!(\mathbb{Z}_V)})) = 0$ $\forall \alpha > \dim Y$

$\Rightarrow H^q(X, \mathbb{K}) = 0 \quad \forall q > \dim X$

| 2*) and
l.e.s.

$\Rightarrow H^q(X, \mathbb{F}) = 0 \quad \forall q > \dim X$

| 3*) l.e.s.

Thus it suff to show | 2*)

4. Step: X irrecl , $\dim X = n \geq 1$
vanishing terms in $\dim < n$

We have to prove:

$$H^q(X, i_! \mathcal{Z}_U) = 0 \quad \forall q > n, \quad \begin{matrix} U \subset X \\ \text{open} \end{matrix}$$

Pf: $U \subset X \supseteq Y = X \setminus U$
 $\begin{matrix} \text{open} \\ \downarrow \\ \text{closed.} \end{matrix}$

$$0 \rightarrow j_! (\mathcal{Z}_U) \rightarrow \mathcal{Z}_X \rightarrow i_! (\mathcal{Z}_Y) \rightarrow 0$$

Assume
| 4*) $H^q(X, \mathcal{Z}_X) = 0 \quad \forall q > n$

then (l.e.s)

$$H^{d-1}(Y, \mathbb{Z}_Y) \rightarrow H^d(X, j_* \mathbb{Z}_Y) \rightarrow H^d(X, \mathbb{Z}_X) \xrightarrow{\text{if } d \geq 1} 0$$

$\stackrel{\text{if } d \geq 1}{\Rightarrow}$

for $d-1 > \dim Y$

important if
 $d > \dim X$
 (since $\dim X-1 \geq \dim Y$)

Thus it suff to show

5. Step: X irrecl \Rightarrow

$$H^d(X, \mathbb{Z}_X) = 0 \quad \forall d \geq 1$$

pf: X irrecl, $U, V \subset X$ open

$$\Rightarrow U \cap V \neq \emptyset$$

$$\Rightarrow \forall U \subset X \text{ open} \quad \mathbb{Z}_X(U) = \mathbb{Z}$$

$\Rightarrow \mathbb{Z}_X$ is flasque $\Rightarrow H^d(X, \mathbb{Z}_X) = 0 \quad \forall d \geq 1$
 §4, (10), i)

□