

§1 (Some) Homological Algebra

Recall :

a category \mathcal{E}

consists of a collection of objects,

we write $x \in \mathcal{E}$ for "x is an object of \mathcal{E} ",

and $\forall x, y$ a set of morphisms

$\text{Hom}_{\mathcal{E}}(x, y)$ with the following datum

(i) $\forall x, y, z \in \mathcal{E} \exists$ composition law

$$\text{Hom}_{\mathcal{E}}(x, y) \times \text{Hom}_{\mathcal{E}}(y, z) \xrightarrow{\circ} \text{Hom}_{\mathcal{E}}(x, z)$$

$$(f, g) \mapsto g \circ f$$

(ii) $\forall x \in \mathcal{E} \exists \text{id}_x \in \text{Hom}_{\mathcal{E}}(x, x)$

such that

a) the composition is associative

b) $\text{id}_y \circ f = f = f \circ \text{id}_x \forall f \in \text{Hom}_{\mathcal{E}}(x, y)$

Let \mathcal{C}, \mathcal{D} be categories

Recall that a covariant functor or just functor $F: \mathcal{C} \rightarrow \mathcal{D}$
 is $\forall x \in \mathcal{C}$ an obj $F(x) \in \mathcal{D}$

and $\forall f: x \rightarrow y$ a morphism $F(f) \in \text{Hom}_{\mathcal{D}}(F(x), F(y))$
 $\in \text{Hom}_{\mathcal{C}}(x, y)$

s.t. $F(f) \circ F(g) = F(f \circ g) \quad \forall f \in \text{Hom}(y, z)$
 $F(\text{id}_x) = \text{id}_{F(x)} \quad \forall x \in \mathcal{C} \quad g \in \text{Hom}(x, y)$

A contravariant functor from \mathcal{C} to \mathcal{D}

is a functor $F: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$

where \mathcal{C}^{op} is the opposite category of \mathcal{C} ,

i.e. $\text{obj}(\mathcal{C}^{\text{op}}) = \text{obj}(\mathcal{C})$

$\text{Hom}_{\mathcal{C}^{\text{op}}}(x, y) := \text{Hom}_{\mathcal{C}}(y, x) \quad \forall x, y \in \mathcal{C}$.

Thus $\forall f: x \rightarrow y$ in \mathcal{C} we get $F(y) \rightarrow F(x)$

and $F(f) \circ F(g) = F(g \circ f)$

① Def. of cat

We say \mathcal{C} is an additive category : \iff

i) $\forall X, Y \in \mathcal{C}$: $\text{Hom}_{\mathcal{C}}(X, Y)$ is an abelian group

s.t. & $f, f' \in \text{Hom}_{\mathcal{C}}(X, Y)$

and $g \in \text{Hom}_{\mathcal{C}}(W, X)$, $h \in \text{Hom}_{\mathcal{C}}(Y, Z)$

$$(f + f') \circ g = f \circ g + f' \circ g, \quad g \circ (f + f') = g \circ f + g \circ f'$$

ii) $\exists 0 \in \mathcal{C}$ zero object, i.e. an object

such that

$$\underset{\mathcal{C}}{\text{Hom}}(X, 0) = \{0\}, \quad \underset{\mathcal{C}}{\text{Hom}}(0, X) = \{0\}$$

\uparrow
zero-group

$\forall x \in \mathcal{C}$

iii) $\forall X, Y \in \mathcal{C}$ a product

$$X \times Y \in \mathcal{C}$$

i.e. an obj with morphisms $X \times Y \rightarrow Y$.

satisfying

$$\begin{array}{ccc} Z & \xrightarrow{\exists !} & X \times Y \\ & \searrow & \downarrow & \swarrow \\ & & X & & Y \end{array}$$

[it is
determined
up to unique
isomorphism.]

③ Rank In an additive category
 the coproduct (direct sum) $X \oplus Y$ also exists
 and is equal to $X + Y$ (see Exercises)

④ Def: A, B add cats

$F: A \rightarrow B$ is an additive functor

: $\Leftrightarrow F: \text{Hom}_A(X, Y) \rightarrow \text{Hom}_B(F(X), F(Y))$
 is a group hom & $X, Y \in A$

similar with contravariant functors.

⑤ Example A add cat, $(Ab) = \text{cat of ab groups}$
 (obj = ab groups
 morph = group hom.)

$X \in A \Rightarrow$

$\text{Hom}_A(X, -) : A \rightarrow (Ab)$ is an additive
 $y \mapsto \text{Hom}_A(X, Y)$ functor

$\text{Hom}_A(-, X) : A \rightarrow (Ab)$
 $y \mapsto \text{Hom}_A(Z, X)$ is a contravariant
 additive functor.

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⑤ Def. $f: X \rightarrow Y$ in add cat A

(i) Kernel of f is (if it exists) an morphism.

$$i: \text{Ker}(f) \rightarrow X \text{ s.t. } fi = 0$$

and

$$\begin{array}{ccc} \text{Ker}(f) & \xrightarrow{i} & X \xrightarrow{f} Y \\ & \nearrow \exists! \quad \searrow \exists! & \nearrow \exists! \\ & Z & \end{array}$$

(ii) Cokernel of f is a morphism

$$Y \xrightarrow{p} \text{coker}(f) \text{ s.t. } pf = 0$$

and

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \xrightarrow{p} \text{coker}(f) \\ & \searrow \exists! & \downarrow \exists! \\ & Z & \end{array}$$

(iii)

f is monic : \Leftrightarrow

$$[f \circ g = 0, \text{ for } g: Z \rightarrow X \Rightarrow g = 0]$$

(iv) f is epi : \Leftrightarrow

$$[h \circ f = 0, \text{ for } h: Y \rightarrow Z \Rightarrow h = 0]$$

⑥ Def.: An abelian category

is an additive category A s.t.

- (i) $\forall f: x \rightarrow y \text{ in } A \quad \exists \text{ Ker}(f) \xrightarrow{i} x$
- (ii) $\forall f: x \rightarrow y \text{ in } A \quad \exists \text{ } y \xrightarrow{p} \text{Coker}(f)$
- (iii) • Every monic is the kernel of its cokernel, i.e.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \text{monic} & \parallel & \\ \text{Ker}(p|f) & \xrightarrow{i(p|f)} & Y \end{array}$$

- Every epi is the cokernel of its kernel, i.e.

$$\begin{array}{ccc} \text{Im}(f) & \xrightarrow{i(f)} & X \xrightarrow{f} Y \\ \parallel & \text{epi} & \parallel \\ X & \xrightarrow{p(f)} & \text{Coker}(i(f)) \end{array}$$

Ques.

— A abelian cat $f: X \rightarrow Y$

$$\text{Im}(f) := \text{image of } f := \text{Ker } (Y \rightarrow \text{Coker}(f))$$

③ Examples: Abelian categories are

- (i) (Ab) = cat of abelian groups
 - obj: ab groups
 - \morph: group hom.
 - (ii) R = commutative ring with 1
 - $(R\text{-mod})$ = category of R -modules
 - obj: R -modules
 - \morph: R -linear homom.
 - (iii) X topological space
 - $\text{Sh}_v(X)$ = category of sheaves of abelian groups on X
 - obj: sheaves of ab gps on X
 - \morph: morphisms of sheaves of ab gps.
- Obj in $\text{Sh}_v(X)$ is $\bigcup_{\text{open } U} \{0\}$
 - product $F, G \in \text{Sh}_v(X) \Rightarrow X \supset U \mapsto (F \times G)(U) := F(U) \times G(U)$
 - $f: F \rightarrow G \in \text{Sh}_v(Y) \Rightarrow \text{Ker}(f)(U) := \text{Ker}(f(U): F(U) \rightarrow G(U))$
free sheaf.

The (also f^*) is the sheafification

of the presheaf

$$X \xrightarrow{\text{open}} U \mapsto \text{coker}(f|_{\mathcal{O}_U}) = \frac{\mathcal{O}_U(U)}{f(F(U))}$$

Thus to give a section

$$s \in T(U, \text{coker}(f))$$

$$\hookrightarrow \exists U = \bigcup_i U_i \text{ open cover}$$

$$\exists \sigma_i \in \mathcal{O}_U(U_i) \text{ s.t.}$$

$$\sigma_i|_{U_i \cap U_j} - \sigma_j|_{U_i \cap U_j} \in f(F(U_i \cap U_j)) \quad \forall i, j$$

$$\text{and } s|_{U_i} = \sigma_i \bmod f(F(U_i))$$

\Rightarrow $\text{Im } f$ is the sheafification of $U \mapsto f(F(U)) \subset \mathcal{O}_U(U)$

(iv) X scheme with structure sheaf \mathcal{O}_X

$\Rightarrow \mathcal{O}_X\text{-Mod} = \text{category of } \mathcal{O}_X\text{-modules}$

$Q\text{coh}(X) = \text{category of quasi-coherent } \mathcal{O}_X\text{-mod}$

$Coh(X) = \text{category of coherent } \mathcal{O}_X\text{-mod.}$

($\mathcal{O}, \oplus, \text{ker, coker}$ defined as in (iii), they naturally are equipped with more str.)

(9) Def: A abelian cat

(ii) A complex in A is a collection of objects $\{C^i\}_{i \in \mathbb{Z}}$ with morphs $\{d^i: C^i \rightarrow C^{i+1}\}_{i \in \mathbb{Z}}$

$$\text{s.t. } d^{i+1} \circ d^i = 0 \quad \forall i \in \mathbb{Z}$$

:

We write

$$\dots \rightarrow C^i \xrightarrow{d^i} C^{i+1} \xrightarrow{d^{i+1}} C^{i+2} \rightarrow \dots = (C, d)$$

$$= C^\bullet$$

(If C^i are only given for $a \leq i \leq b$
 then we extend via $C^i = 0 \wedge i < a \text{ or } i > b$)

(ii) $C^\bullet = (C, d)$ a complex

the i-th cohomology group of C^\bullet is given by

$$H^i(C^\bullet) = \frac{\text{Ker}(d^i: C^i \rightarrow C^{i+1})}{\text{Im}(d^{i-1}: C^{i-1} \rightarrow C^i)}$$

↑ Note C^\bullet complex \Rightarrow

↑

quation

$$\text{Im } d^{i-1} \subset \text{Ker } d^i$$

↓

in A
 i.e. coker of

L

(iii) We say C^\cdot is acyclic $\iff H^i(C^\cdot) = 0 \forall i$

(10) Ex:

$$\cdots \rightarrow \frac{\mathbb{Z}}{6\mathbb{Z}} \xrightarrow{[3]} \frac{\mathbb{Z}}{6\mathbb{Z}} \xrightarrow{[2]} \frac{\mathbb{Z}}{6\mathbb{Z}} \xrightarrow{[3]} \frac{\mathbb{Z}}{6\mathbb{Z}} \xrightarrow{[2]} \frac{\mathbb{Z}/6\mathbb{Z}}{C^1} \rightarrow \frac{\mathbb{Z}/2\mathbb{Z}}{C^0} \rightarrow 0$$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$C^{-5} \quad C^{-4} \quad C^{-3} \quad C^{-2} \quad C^{-1} \quad C^0$

$\Rightarrow C^\cdot$ is an acyclic complex

(11) Def. We say a sequence

$$C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} C^3 \xrightarrow{d^3} \cdots \xrightarrow{d^{n-1}} C^n$$

in an ab cat is exact

$$\Leftrightarrow \ker(d^i) = \text{Im}(d^{i-1})$$

$$\text{And } 2 \leq i \leq n-1$$

* A short exact sequence is an exact sequence

$$0 \rightarrow A \xrightarrow{a} B \xrightarrow{b} C \rightarrow 0 \quad \text{i.e. } a \text{ is monic}$$

b is epi
 $\ker b = \text{Im } a$

(12) Proposition (Snake Lemma, module version)

Let R be a ring with 1 (not nec commutative)

Let

$$\begin{array}{ccccccc} & M' & \xrightarrow{a} & M & \xrightarrow{b} & M'' & \rightarrow 0 \\ f \downarrow & & & \downarrow f & & \downarrow f'' & \\ 0 \rightarrow N' & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & N'' & & \end{array}$$

be a commutative diagram of left R -modules
with exact rows

$$(i.e., f \circ a = \alpha \circ f', f'' \circ b = \beta \circ f)$$

b is surj, α is inj $\text{Im}(b) = \text{Im}(\alpha)$, $\text{Ker}(\beta) = \text{Im}(\alpha)$

Then \exists an exact seq

$$\begin{array}{ccccccc} \text{if } \alpha \text{ is inj} & & & & & & \\ 0 \rightarrow & \text{Ker } f' & \xrightarrow{a^0} & \text{Ker } f & \xrightarrow{b^0} & \text{Ker } f'' & \\ \downarrow & & & & & & \end{array}$$

$$\begin{array}{ccccccc} & & & & & & \\ S & \swarrow & \text{coker } f' & \xrightarrow{\bar{\alpha}} & \text{coker } f & \xrightarrow{\bar{\beta}} & \text{coker } f'' \dots \rightarrow 0 \\ & & & & & & \\ & & & & & & \text{if } \beta \text{ is surj.} \end{array}$$

where $a^0 = a|_{\text{Ker } f'}$, $b^0 = b|_{\text{Ker } f}$, $\bar{\alpha}, \bar{\beta}$ are induced by α, β and

δ is defined as follows : (*)

$$m'' \in \text{Ker}(f'') \subset M'' \Rightarrow \exists m \in M : b(m) = m''$$

$$\text{and } \beta f(m) = f'' b(m) = f''(m'') = 0$$

$$\stackrel{\Rightarrow}{\text{exactness}} \exists! n' \in N' \text{ s.t. } \alpha(n') = f(m)$$

Then set $\delta(m'') := \overline{n'} \equiv n' \pmod{f'(M')}$

$$\in (\text{oker } f')$$

Bew: a°, b° well-defined, i.e.

$$\alpha(\text{Ker}(f')) \subset \text{Ker } f, \quad b(\text{Ker}(f)) \subset \text{Ker } f'':$$

follows from commutativity of the diagram .

$\bar{\alpha}, \bar{\beta}$ well defined, i.e.

$$\alpha(\text{Im}(f'')) \subset \text{Im}(f), \quad \beta(\text{Im}(f)) \subset \text{Im } f'':$$

(Then $\exists : (\text{oker } f' = \frac{N'}{\text{Im}(f')}) \rightarrow \frac{N}{\text{Im}(f)} = (\text{oker } f \text{ well-defined similar with } \beta)$)

follows from commutativity of the diagram .

S well-defined: Have to show that

(i) $S(m'')$ as defined in (*) is independent of the choice of $m \in M$ with $b(m) = m''$

(ii) S is \mathbb{R} -linear

(ii): Take $m, \tilde{m} \in M$ with $b(m) = m'' = b(\tilde{m})$

$$\Rightarrow b(m - \tilde{m}) = 0 \Rightarrow \exists m' \in M \text{ s.t.}$$

$$m - \tilde{m} = a(m')$$

$$\Leftrightarrow m = \tilde{m} + a(m')$$

$$\Rightarrow f(m) = f(\tilde{m}) + f(a(m')) = f(\tilde{m}) + \alpha f'(m')$$

$$\text{As in (*) } \exists! \tilde{m}' \in N^1 \text{ s.t. } \alpha(\tilde{m}') = f(\tilde{m})$$

$$\Rightarrow f(m) = \alpha(\underbrace{\tilde{m}' + f'(m')}_=: n')$$

$$\Rightarrow \tilde{m}' = \tilde{m} \bmod f'(m')$$

$\Rightarrow S$ well-def.

(iii): $m_i'' \in M''$ $i=1,2$, $\lambda \in \mathbb{R}$, choose $m_i \in M$: $b(m_i) = m_i''$

$$\Rightarrow \exists! n_i \in N^1 \text{ s.t. } f(m_i) = \alpha(n_i)$$

$$\begin{aligned} \Rightarrow S(m_1'') + \lambda S(m_2'') &= \overline{\tilde{m}_1' + \lambda \tilde{m}_2'} \in \text{ker } f' \\ &= S(m_1'' + \lambda m_2'') \end{aligned}$$

* a^0 inj if a is:

$$\begin{array}{ccc} \text{Ker } f' & \xrightarrow{a^0} & \text{Ker } f \\ \downarrow & & \downarrow \\ M' & \xhookrightarrow{a} & M \end{array}$$

OK

* $\text{Im } a^0 = \text{Ker } b^0$: $i : \text{Ker } f'' \hookrightarrow M''$

$$z \circ b^0 \circ a^0 = b \circ a |_{\text{Ker } f''} = 0 \Rightarrow b^0 \circ a^0 = 0$$

$$\Rightarrow \text{Im } a^0 \subset \text{Ker } b^0$$

2. Let $m \in \text{Ker } f$ with $b(m) = 0$

$\stackrel{\text{defn}}{\Rightarrow} \exists m' \in M' \text{ s.t. } m = a(m')$

and $a f'(m') = f a(m') = f(m) = 0$

$\stackrel{\alpha \text{ inj}}{\Rightarrow} f'(m') = 0$, i.e., $m' \in \text{Ker } f'$ and $a^0(m') = m \in \text{Im } a^0$

* $\text{Im } b^0 = \text{Ker } f$:

\subseteq : $m'' \in \text{Im } (b^0)$ i.e. $\exists m \in \text{Ker } f : b(m) = m''$

$\Rightarrow f(m) = 0 = \alpha(0) \Rightarrow \delta(m'') = 0$

\supseteq : $m'' \in \text{Ker } f'$, $m \in M$ s.t. $b(m) = m''$

$\exists n' \in N' \text{ s.t. } \alpha(n') = m$

Assume $\delta(m'') = \bar{n}' = 0$ in $\text{CoKer } f'$

$$\Rightarrow \exists x' \in M' \text{ s.t. } m' = f'(x')$$

$$\Rightarrow f(m) = \alpha(m') = \alpha(f(x)) = f(\alpha(x))$$

$$\Rightarrow m - \alpha(x') \in \ker(f) \text{ and}$$

$$b^0(m - \alpha(x')) = b(m) = m''$$

$$\text{i.e. } m'' \in \operatorname{Im} b^0 \Rightarrow \ker \delta \subset \operatorname{Im} b^0$$

$$\operatorname{Im} \delta = \overline{\ker \bar{\delta}}$$

$$\underline{c:} \quad m'' \in \ker f'', \quad b(m) = m'', \quad \alpha(m') = f(m)$$

$$\Rightarrow \delta(m'') = \bar{m}' \in \operatorname{Coker} f'$$

$$\text{and } \bar{\delta} \delta(m'') = \overline{\delta(m')} = \overline{f(m)} = 0 \in \operatorname{Coker} f$$

$$\underline{d:} \quad \text{let } m' \in N' \text{ s.t. } \bar{\delta}(\bar{m}') = \overline{\delta(m')} = 0 \\ \text{in } \operatorname{Coker} f$$

$$\Rightarrow \alpha(m') = f(m) \text{ for some } m \in M$$

$$\Rightarrow \bar{m}' = \delta(b(m)) \in \operatorname{Im} H.$$

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$\text{Im } \bar{\alpha} = \text{Ker } \bar{\beta} :$

$$\bar{\beta} \circ \bar{\alpha} = \overline{\beta \circ \alpha} = 0 \Rightarrow \text{Im } \bar{\alpha} \subset \text{Ker } \bar{\beta}$$

$\supset:$ $n \in N$ with $\bar{\beta}(\bar{n}) = \overline{f(n)} = 0$
in $\text{Ker } f''$

$$\Rightarrow \exists n'' \in M'': \beta(n) = f''(n'')$$

$\xrightarrow{b \text{ surj}}$ $\exists m \in M$ s.t. $b(m) = n''$

$$\Rightarrow 0 = f''(n'' - b(m)) = \beta(m - f(m))$$

$\xrightarrow{\text{Ker } \beta = \text{Im } \alpha}$ $\exists n' \in N'$ s.t.

$$m - f(m) = \alpha(n')$$

$\xrightarrow{\text{mod } f(M)}$ $\bar{n} = \bar{\beta}(\bar{n}')$ in $\text{Ker } f$
 $\in \text{Im } \bar{\alpha}$

If β is surj, then $\bar{\beta}$ surj:

$$\begin{array}{ccc}
 N & \xrightarrow{\beta} & N'' \\
 \downarrow & & \downarrow \\
 \frac{N}{f(M)} & \xrightarrow{\bar{\beta}} & \frac{N''}{f''(M'')}
 \end{array}
 \rightarrow \text{OK}$$

□

(13) Prop (Snake Lemma) A abelian cat

$$\begin{array}{ccccccc} C' & \rightarrow & C & \rightarrow & C'' & \rightarrow & 0 \\ \downarrow f & & \downarrow f & & \downarrow f'' & & \\ 0 & \rightarrow & D' & \rightarrow & D & \rightarrow & D'' \end{array} \quad \text{commutes with exact rows}$$

$\Rightarrow \exists$ (canonical exact sequence)

PF: In the examples from (8) a similar proof as in (7) works.

One can also reduce the general case to the case of R -modules using the Freyd-Mitchell Embedding Theorem.

(see e.g. Weibel, Introduction to Homological algebra,
Thm 1.G.1)

Term 1.G.1 |

四

⑭ Def: $(C, \delta), (D, \delta)$ complexes in A

i) A morphism $C \xrightarrow{\varphi} D$ of complexes

is a collection $\varphi = \{\varphi^i\}_{i \in \mathbb{Z}}$ where

$\varphi^i: C^i \rightarrow D^i$ is a morphism in A

s.t. $\forall i$ the diagram

$$\begin{array}{ccc} C^i & \xrightarrow{\varphi^i} & D^i \\ d \downarrow & & \downarrow \delta^i \\ C^{i+1} & \xrightarrow{\varphi^{i+1}} & D^{i+1} \end{array}$$

commutes i.e.

$$\delta^i \circ \varphi^i = \varphi^{i+1} \circ d^i$$

→ category of complexes in A

$\mathcal{Ch}(A)$ (Ex: $\mathcal{Ch}(A)$ is an abelian cat)

i) We say a sequence

$$0 \rightarrow C \xrightarrow{\varphi} D \xrightarrow{\psi} E \rightarrow 0$$

in $\text{Ch}(A)$ is a short exact sequence
of complexes

\Leftrightarrow

$$\forall i \quad 0 \rightarrow C^i \xrightarrow{\varphi^i} D^i \xrightarrow{\psi^i} E^i \rightarrow 0$$

is a short exact sequence in A

(15) Prop: Let A be an abelian cat

$0 \rightarrow C \xrightarrow{\varphi} D \xrightarrow{\psi} E \rightarrow 0$ be a short exact seq in $\text{Ch}(A)$

Then there is a long exact sequence of cohomology groups

$$\dots \rightarrow H^{i-1}(I) \xrightarrow{\delta} H^i(C) \xrightarrow{H^i(\varphi)} H^i(D) \xrightarrow{H^i(\psi)} H^i(E) \xrightarrow{\partial} H^{i+1}(C) \xrightarrow{H^{i+1}(\varphi)} H^{i+1}(D) \rightarrow \dots$$

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Proof:

$$\text{Set } Z^i(C) = \ker(d^i : C^i \rightarrow C^{i+1}) \quad (\text{also for } D, I)$$

$$B^i(C) = \text{Im}(d^{i-1} : C^{i-1} \rightarrow C^i) \quad (\text{also for } D, I)$$

We have

$$0 \rightarrow B^i(C) \rightarrow Z^i(C) \rightarrow H^i(C) \rightarrow 0$$

Short ex seq.

Have \mathbb{H}^i we have a commutative diagram
with exact rows

$$\begin{array}{ccccccc} 0 & \rightarrow & C^i & \xrightarrow{\quad} & D^i & \xrightarrow{\quad} & E^i \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & C^{i+1} & \xrightarrow{\quad} & D^{i+1} & \xrightarrow{\quad} & E^{i+1} \rightarrow 0 \end{array}$$

\Rightarrow get commutative diagram

$$\begin{array}{ccccc} C^i & \xrightarrow{\quad} & D^i & \xrightarrow{\quad} & E^i \\ \overline{B^i(C)} & \xrightarrow{\quad} & \overline{B^i(D)} & \xrightarrow{\quad} & \overline{B^i(E)} \\ \downarrow d_C & & \downarrow d_D & & \downarrow d_E \\ 0 \rightarrow Z^{i+1}(C) & \rightarrow & Z^{i+1}(D) & \rightarrow & Z^{i+1}(E) \end{array}$$

(rows are exact
by Smith Lemma)

\Rightarrow

Smith Lemma

$$\begin{array}{ccccccc} (\ker d_C \rightarrow \ker d_D \rightarrow \ker d_E \xrightarrow{\exists} \text{ker } d_C \rightarrow \text{ker } d_D \rightarrow \text{ker } d_E) \\ \parallel \qquad \parallel \qquad \parallel \\ H^i(C) \qquad H^i(D) \qquad H^i(E) \end{array}$$

$$H^{i+1}(C)$$

$$H^{i+1}(D)$$

$$H^{i+1}(E)$$

□

(16) Def:

(1) let $\mathcal{F} : \mathcal{A} \rightarrow \mathcal{B}$ be

a (covariant) functor between
abelian categories

We say

\mathcal{F} is $\left\{ \begin{array}{l} \text{exact} \\ \text{left-exact} \end{array} \right\}$

$\Leftrightarrow \mathcal{F}$ short exact sequences in \mathcal{A}

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

the sequence $\left\{ \begin{array}{l} 0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(Y) \rightarrow \mathcal{F}(Z) \rightarrow 0 \\ 0 \rightarrow \mathcal{F}(X) \rightarrow \mathcal{F}(Y) \rightarrow \mathcal{F}(Z) \end{array} \right\}$

is exact in \mathcal{B}

(2) let $G : A \rightarrow B$ be a contravariant functor

(i.e. $G : A^{\text{op}} \rightarrow B$ is a functor)

Then we say

G is $\left\{ \begin{array}{l} \text{exact} \\ \text{left exact} \end{array} \right\} \iff$

A short exact sequences in A

$$0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$$

The sequence

$$\left\{ \begin{array}{l} 0 \rightarrow G(Z) \rightarrow G(Y) \rightarrow G(X) \rightarrow 0 \\ 0 \rightarrow G(Z) \rightarrow G(Y) \rightarrow G(X) \end{array} \right\}$$

is exact in B

(17) Prop.: Let A be an abelian category

$X \in A$

Then $\text{Hom}(-, X) : A \rightarrow (\text{ab groups})$
 $y \mapsto \text{Hom}_A(y, X)$

is a left exact contravariant functor.

Pf.:

• A additive cat $\Rightarrow \text{Hom}(y, X) \in (\text{ab groups})$
 $\forall y$

• contravariant functor:

$y' \xrightarrow{f} y$ in A

$\Rightarrow f^* := \text{Hom}(f, 1) : \text{Hom}(y, X) \rightarrow \text{Hom}(y', X)$
 $g \mapsto f^*(g) := g \circ f$

clearly: $g^* f^* = (f \circ g)^*$ $\quad \forall f \in \text{Hom}(y', y)$
 $h^* = \text{id}_{\text{Hom}(y, X)}$ $\quad \forall h \in \text{Hom}(y'', y')$

group form: $f^*(g + g') = (g + g') \circ f = \underset{\substack{\uparrow \\ \text{add cat}}}{g \circ f + g' \circ f} = f^*g + f^*g'$

• left exact:

$$0 \rightarrow Y' \xrightarrow{f} Y \xrightarrow{g} Y'' \rightarrow 0 \quad \text{s.e.s. in } A$$

\rightarrow get sequence

$$0 \rightarrow \text{Hom}(Y'', X) \xrightarrow{f^*} \text{Hom}(Y, X) \xrightarrow{g^*} \text{Hom}(Y', X)$$

complex since $f^* g^* = (g \circ f)^* = 0$

$$\Rightarrow \text{Im } g^* \subset \text{Ker } f^*$$

$\text{Ker } f^* \subset \text{Im } g^*:$ $s \in \text{Ker } f^*$

let

$$\begin{array}{ccc} Y & \xrightarrow{g} & X \\ f \uparrow & \nearrow & \\ Y' & \xrightarrow{f^* g = 0} & \end{array}$$

s.e.s

\Rightarrow
def of Ker

$$\begin{array}{ccccccc} 0 \rightarrow & Y' & \xrightarrow{f} & Y & \xrightarrow{g} & \text{coker } f \cong Y'' & \downarrow \\ & 0 \downarrow & \downarrow & & \ddots & \ddots & \\ & X & \xrightarrow{\tilde{f}} & & & \tilde{f} & \end{array}$$

and $h^* \tilde{f} = s \Rightarrow s \in \text{Im } h^*$

$\text{Im } g^* = 0:$

$g \in \text{Im } g^* \Rightarrow$

$$\begin{array}{ccccc} 0 & \xrightarrow{\text{id}_Y} & Y' & \xrightarrow{g} & Y'' \rightarrow 0 = \text{coker } g \\ & \xrightarrow{\text{id}_Y^*} & \downarrow g & & \\ & 0 & \xrightarrow{\text{id}_X} & X & \end{array} \Rightarrow g = 0$$

□

(18) Rank:

$\text{Hom}(-, X)$ is in general not exact

Ex:

$$0 \rightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0 \quad \text{exact}$$

apply $\text{Hom}(-, \mathbb{Z})$

$$\Rightarrow 0 \rightarrow \text{Hom}(\mathbb{Z}/2, \mathbb{Z}) \rightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot 2} \text{Hom}(\mathbb{Z}, \mathbb{Z})$$

$\parallel \quad \parallel \quad \parallel$
 $0 \quad \mathbb{Z} \quad \mathbb{Z}$
↓
not surj.

