

# Cohomology of sheaves on schemes

## § 0 Motivation

$k = \text{field}$

1.  $X/k$  variety, i.e. scheme of fin type and  $\text{sep}/k$

Assume  $X$  is somehow abstractly given.

We can try to study  $X$  by embedding it  
in to some projective space  $\mathbb{P}_k^3$

( $\mapsto$  give  $X$  by equations)

or at least by finding a map  $X \xrightarrow{f} \mathbb{P}_k^3$

Such maps  $f$  correspond to

$(L, s_0, \dots, s_n)$  where  $L$  is an invertible sheaf  
on  $X$  (i.e.  $L$  is a loc free  $\mathcal{O}_X$ -mod  
of rank 1)

and  $s_0, \dots, s_n \in H^0(X, L)$  are

global sections generating  $L$ , i.e.

$\bigoplus_{i=0}^n \mathcal{O}_X \rightarrow L, (a_i) \mapsto \sum a_i s_i$  is surj.

⇒ Thus to study  $X$  we might try

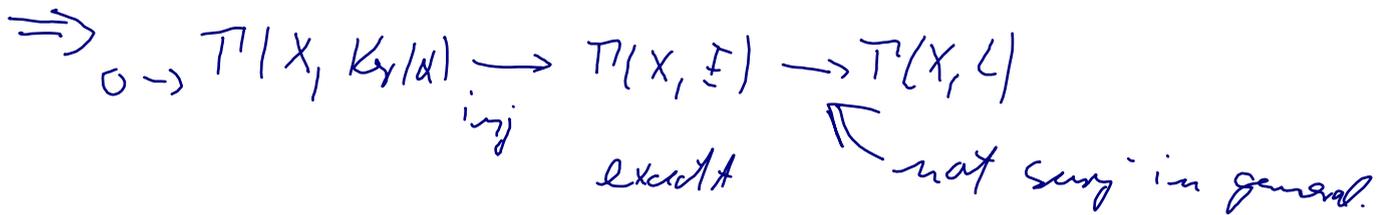
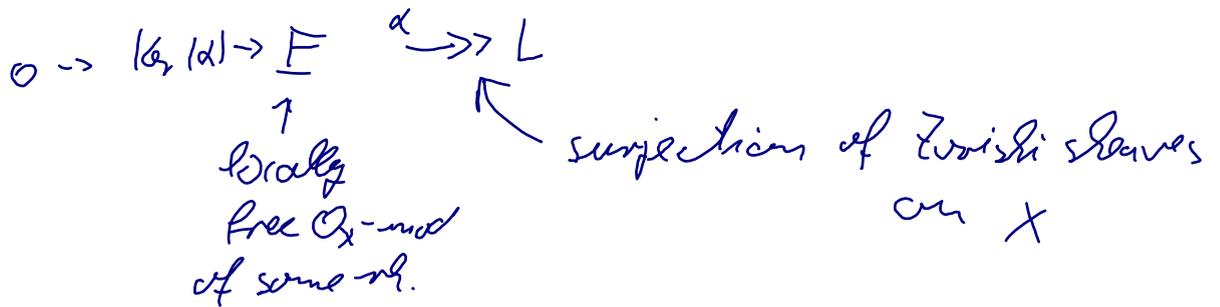
to compute  $\dim_x T(X, L)$

at least when this number is finite.

⇒ (i) When is  $\dim_x T(X, L)$  finite?

(ii) How to compute it?

Often  $L$  is constructed as a quotient



Understanding of  $k_r(d)$  and  $\underline{F}$

should lead to an understanding of  $T(X, L)$ .

Solution: cohomology yields exact sequence

$$\begin{aligned} \Gamma(X, \mathcal{O}_X(d)) &\rightarrow \Gamma(X, \mathcal{E}) \hookrightarrow \Gamma(X, \mathcal{L}) \rightarrow H^1(X, \mathcal{O}_X(d)) \\ &\rightarrow H^1(X, \mathcal{E}) \rightarrow H^1(X, \mathcal{L}) \rightarrow H^2(X, \mathcal{O}_X(d)) \end{aligned}$$

"long exact cohomology sequence"

2. Problem Decide whether two given varieties are isomorphic to each other.

→ difficult to check.

Ex:  $\mathbb{A}^2 = \overline{\mathbb{A}^2}$ , or  $\mathbb{A}^2 \neq \mathbb{A}^2$

$$C = V(aX^2 + bY^2 + cZ^2 + dXY + eXZ + fYZ) \subset \mathbb{P}_{\mathbb{A}^2}^2$$

with  $\det \begin{pmatrix} a & \frac{d}{2} & \frac{e}{2} \\ \frac{d}{2} & b & \frac{f}{2} \\ \frac{e}{2} & \frac{f}{2} & c \end{pmatrix} \neq 0$

$$\Rightarrow C \cong \mathbb{P}_{\mathbb{A}^2}^1$$

Consider

$$C_1 = V(F(x, y, z))$$

with  $F(x, y, z) \in \mathcal{S}[x, y, z]$

homogeneous of degree  $d \geq 3$

Can it happen that

$$C_1 \cong \mathbb{P}^1 ?$$

No - never!

Since  $X \rightarrow H^1(X, \mathcal{O}_X)$  preserves isomorphisms

$\uparrow$  proj. 2-variety  $\uparrow$  finite dim'l

1st cohomology group

and  $H^1(C_1, \mathcal{O}_{C_1}) \neq H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$

---

In general projective varieties can be (partially) classified by cohomological invariants such as  $H^i(X, \mathcal{O}_X)$ ,  $i \geq 0$ , etc. ....