Some solutions for Exercise sheet 11 [1]

Here are the solutions for Exercise 11.1, (iii) and Exercise 11.2, which I could not do in class today. They are a little bit more involved than I thought, sorry!

Exercise 11.1, (iii): Let R be a PID and M a finitely generated torsion-free R-module of rank n. Show that there exists an exact sequence of finitely generated torsion-free R-modules $0 \to N \to M \to M/N \to 0$, where $\mathrm{rank}(N) = n-1$ and $\mathrm{rank}(M/N) = 1$.

Solution: Set $K := \mathrm{Frac}(R)$ and denote by $\varphi : M \to M \otimes_R K$ the injection given by $\varphi(m) = m \otimes 1$. By part (ii) of the exercise we find $m_1 \ldots, m_n \in M$ such that $\varphi(m_1), \ldots, \varphi(m_n)$ form a basis of $M \otimes_R K$. Denote by $N' \subset M \otimes_R K$ the K-subvector space generated by $\varphi(m_1), \ldots, \varphi(m_{n-1})$. Set $N := \varphi^{-1}(N')$. It is an R-submodule of M and hence is torsion-free.

Next we claim that N is finitely generated as R-module. [Indeed, we prove that if R is a PID and M is a finitely generated R-module, then any submodule $N \subset M$ is finitely generated. To this end let $x_1, \ldots, x_r$ be a set of generators of M. We do induction over r. If $r = 1$, then we have a surjection $\alpha : R \to M$, $a \mapsto ax_1$. Now $\alpha^{-1}(N)$ is an R-submodule of R, hence is an ideal, hence is generated by one element (R is a PID); it follows that N is generated by one element as well. Assume the statement is true for $r-1$ generators. Write $M' = \sum_{i=1}^{r-1} R \cdot x_i$ and $M'' = M/M'$. Then M' and M'' are generated by $\leq r-1$ elements. Set $N' := M' \cap N$ and $N'' := N/N'$. Then $N' \subset M'$ and $N'' \subset M''$. By induction N' and N'' are finitely generated, hence so is N as follows from the short exact sequence $0 \to N' \to N \to N'' \to 0$. Back to the proof of Exc 11.1,(iii):]

By definition $N = \{m \in M \mid \exists\, a \in R \text{ such that } am \in \sum_{i=1}^{n-1} R \cdot m_i\}$. It follows that $N \otimes_R K = N'$. Thus all together N is a finitely generated torsion-free submodule of M of rank $\mathrm{rk}(N) = n-1$.

Notice that if $m \in M$ and $a \in R$ such that $am \in N$, then $m \in N$. Hence M/N is torsion-free. It is finitely generated, since M is. Since K is flat over R we have an exact sequence $0 \to N \otimes_R K \to M \otimes_R K \to M/N \otimes_R K \to 0$. It follows that the K-vector space dimension of

[1] Questions or comments to kay.ruelling@fu-berlin.de or come to 1.103(RUD25) on Tue/Thu/Fri.

$M/N \otimes_R K$ is 1, i.e. $\mathrm{rk}(M/N) = 1$. This proves (iii).

Exercise 11.2: Show that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module but is not projective.

Solution: $\mathbb{Q} = S^{-1}\mathbb{Z}$ with $S = \mathbb{Z} \setminus \{0\}$. Hence $\mathbb{Q}$ is flat over $\mathbb{Z}$. Next we prove that it is not projective. Consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$, which by definition is the cokernel of the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$. For $n \in \mathbb{N}_{\geq 1}$ denote by $\alpha_n : \mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ the unique $\mathbb{Z}$-module homomorphism with $\alpha_n(1) \equiv \frac{1}{n} \bmod \mathbb{Z}$. Clearly, $\mathrm{Ker}(\alpha_n) = n\mathbb{Z}$; hence α_n factors via $\bar{\alpha}_n : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. By the UMP of the direct sum we get a $\mathbb{Z}$-linear map $\alpha := \oplus_n \alpha_n : \bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. Clearly α is surjective. Denote by $\pi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ the quotient map.

$$\begin{array}{ccccc} & & \mathbb{Q} & & \\ & & \downarrow{\scriptstyle \pi} & & \\ \bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z} & \xrightarrow{\alpha} & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0. \end{array}$$

If $\mathbb{Q}$ was a projective $\mathbb{Z}$-module, then there would exist a map $\pi_1 : \mathbb{Q} \to \bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$, such that $\alpha \circ \pi_1 = \pi$. But any $\mathbb{Z}$-linear homomorphism $\beta : \mathbb{Q} \to \mathbb{Z}/n\mathbb{Z}$ is the zero map. (Since $\beta(\frac{a}{b}) = \beta(\frac{na}{nb}) = n\beta(\frac{a}{nb}) = 0$.) It follows that if $\pi_1 : \mathbb{Q} \to \bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$ is a $\mathbb{Z}$-linear map, then its composition with any projection map $\bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is zero; hence π_1 is zero. Thus $\alpha \circ \pi_1 = \pi$ would imply $\pi = 0$, i.e. $\mathbb{Q}/\mathbb{Z} = 0$, i.e. the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is surjective, which is absurd. Hence $\mathbb{Q}$ is not projective as an $\mathbb{Z}$-module.