In the exercise session we wanted to find a prime ideal in an infinite product of domains, which is not the inverse image of a prime ideal under the projection to one of the factors (in contrast to what happens for finite products). Actually it is a bit more involved than my first try. Here is the example:

We first need the notion of an ultrafilter: Let Λ be a set. Then an ultrafilter on Λ is a subset \( U \subseteq \mathcal{P}(\Lambda) \) of the power set of Λ which satisfies the following properties:

(i) \( \emptyset \notin U \)
(ii) \( A \subseteq B \subseteq \Lambda \) and \( A \in U \implies B \in U \)
(iii) \( A, B \in U \implies A \cap B \in U \)
(iv) \( A \subseteq \Lambda \implies \) either \( A \in U \) or \( \Lambda \setminus A \in U \)

Observe that

(1) \[ A \cup B \in U \implies A \in U \text{ or } B \in U. \]

(Indeed, if \( A \notin U \), then \( \Lambda \setminus A \in U \); hence \((A \cup B) \cap (\Lambda \setminus A) \in U \) but it is a subset set of \( B \); hence \( B \in U \).) It follows that if \( U \) contains a finite set, then it contains a set with one element say \( \{\lambda_0\} \in U \). Then

(2) \[ U = \{A \subseteq \Lambda \mid \lambda_0 \in A\}. \]

(Indeed, if \( \lambda_0 \in A \), then \( A \in U \), by (ii); if \( \lambda_0 \notin A \), then \( A \subseteq \Lambda \setminus \{\lambda_0\} \) and hence \( A \notin U \) by (ii), (i).) Ultrafilters of type (2) are called principal ultrafilters and we just saw that an ultrafilter is principal if and only if it contains a finite set. It can be shown using Zorn’s Lemma that if \( \Lambda \) is an infinite set, then non-principal ultrafilters on \( \Lambda \) exist.

**Claim 1.** Let \( \Lambda \) be an infinite set and \( R_i, i \in \Lambda \), domains. Set \( R = \prod_{i \in \Lambda} R_i \). Let \( U \) be a non-principal ultrafilter on \( \Lambda \) and set

\[ p := \{(a_i)_{i \in \Lambda} \mid \exists A \in U \text{ s. t. } a_i = 0 \text{ if and only if } i \in A \} \subseteq R. \]

Then \( p \subset R \) is a prime ideal, which is not of the form \( \pi_i^{-1}(p_i) \), for some \( i \in \Lambda \) and \( p_i \subset R_i \), where \( \pi_i : R \to R_i \) is the \( i \)-th projection.

**Proof.** For \( \alpha = (a_i) \in R \) define \( A(\alpha) := \{i \in \Lambda \mid a_i = 0\}. \) Then \( \alpha \in p \) if and only if \( A(\alpha) \notin U \).

\( p \) is an ideal: Given \( \alpha, \beta \in p \), \( x \in R \), we have \( A(\alpha) \cap A(\beta) \subset A(\alpha + x\beta) \). Hence \( A(\alpha + x\beta) \in U \), by (ii), (iii), i.e., \( \alpha + x\beta \in p \).

\( p \) is a prime ideal: Take \( \alpha, \beta \in R \) with \( \alpha \beta \in p \). Then \( A(\alpha \beta) \in U \). Since the \( R_i \)’s are domains we have \( A(\alpha \beta) = A(\alpha) \cup A(\beta) \), whence \( A(\alpha) \in U \) or \( A(\beta) \in U \), by (1). It follows that \( \alpha \in p \) or \( \beta \in p \), i.e., \( p \) is prime.

\( p \) is not of the form \( \pi_i^{-1}(p_i) \): Notice that e.g. the element \((\delta'_{i,j})_{i \in \Lambda} \in \pi_j^{-1}(p_j) \), where \( \delta'_{i,j} = 1 \), if \( i \neq j \), and = 0, if \( i = j \). But this element is
not in $p$, since $U$ is a non-principal ultrafilter and hence any element $(a_i) \in p$ has to have infinitely many $a_i = 0$. □