I messed up a bit the proof of the last corollary in the lecture on Thursday, February 16. Sorry for this. Here is the correct argument.

**Corollary 9.** Let $R$ be a Noetherian ring. Then

$$\dim(R[X]) = \dim R + 1.$$  

**Proof.** $\geq$: A strictly increasing chain of prime ideals in $R$

$$p_0 \subsetneq \ldots \subsetneq p_r$$
gives rise to the strictly increasing chain of prime ideals in $R[X]$ 

$$p_0 R[X] \subsetneq \ldots \subsetneq p_r R[X] \subsetneq p_r R[X] + (X).$$

Hence $\dim(R[X]) \geq \dim R + 1$.

$\leq$: Take $q \subset R[X]$ be a prime ideal and set $p := q \cap R$. We have a natural ring homomorphism $R \rightarrow R[X]_q$. We have $R \setminus p \subset R[X] \setminus q$, hence we get an induced map $\varphi : R_p \rightarrow R[X]_q$. [Actually this is flat, we don’t need this, but : by §17, Prop 3, it suffices to show that the localization of $\varphi$ in the one maximal ideal of $R_p$ is flat; to this end observe that we have an inclusion of multiplicative sets $S := R[X] \setminus q \subset T := R[X] \setminus p$ and by §16, Prop. 8, we have $T^{-1}(R[X]_q) = T^{-1}S^{-1}(R[X]) = T^{-1}(R[X])$. Thus $T^{-1}(\varphi)$ is equal to the localization of the natural inclusion $R \hookrightarrow R[X]$; since $R[X]$ is a free $R$ module the latter is flat and hence so is $T^{-1}(\varphi)$ and also $\varphi$.] By §23, Thm 8 we get

$$\begin{align*}
\dim(R[X]_q) &\leq \dim R_p + \dim \left( \frac{R[X]_q}{p R[X]_q} \right). \\
\end{align*}$$

Now set $S := R \setminus p$ and $T := R[X] \setminus q$. Then $S$ and $T$ are multiplicative subsets of $R[X]$ and by definition of $p$ we have $S \subset T$. We have $S^{-1}(R[X]) \cong R_p[X]$ (you can check this directly by hand or using the UMP). By §16, Prop 8

$$R[X]_q = T^{-1}(R[X]) \cong T^{-1}(S^{-1}(R[X])) \cong T^{-1}(R_p[X]).$$

By §16, Cor 6 we get

$$\frac{R[X]_q}{p R[X]_q} \cong T^{-1} \left( \left( \frac{R_p[X]}{p R_p[X]} \right) \right) \cong T^{-1} \left( \frac{R_p}{p R_p} \right).$$

By §16, Cor 10 we have

$$K := \text{Frac}(R/p) \cong R_p/p R_p.$$  

Thus all together

$$\frac{R[X]_q}{p R[X]_q} = T^{-1}(K[X]) = K[X]_q.$$
Plugging this into (*) we get 
\[ \dim(R[X]_q) \leq \dim R_p + \dim K[X]_q \leq \dim R_p + 1 \leq \dim R + 1. \]
Hence 
\[ \dim R[X] = \sup_q (\dim(R[X]_q)) \leq \dim R + 1. \]
This proves the statement. \qed