I messed up a bit the proof of the last corollary in the lecture on Thursday, February 16. Sorry for this. Here is the correct argument.

Corollary 9. *Let R be a Noetherian ring. Then*

$$\dim(R[X]) = \dim R + 1.$$

Proof. $\geq$: A strictly increasing chain of prime ideals in R

$$\mathfrak{p}_0 \subsetneq \ldots \subsetneq \mathfrak{p}_r$$

gives rise to the strictly increasing chain of prime ideals in $R[X]$

$$\mathfrak{p}_0 R[X] \subsetneq \ldots \subsetneq \mathfrak{p}_r R[X] \subsetneq \mathfrak{p}_r R[X] + (X).$$

Hence $\dim(R[X]) \geq \dim R + 1$.

$\leq$: Take $\mathfrak{q} \subset R[X]$ be a prime ideal and set $\mathfrak{p} := \mathfrak{q} \cap R$. We have a natural ring homomorphism $R \to R[X]_\mathfrak{q}$. We have $R \setminus \mathfrak{p} \subset R[X] \setminus \mathfrak{q}$, hence we get an induced map $\varphi : R_\mathfrak{p} \to R[X]_\mathfrak{q}$. [Actually this is flat, we don't need this, but : by §17, Prop 3, it suffices to show that the localization of φ in the one maximal ideal of $R_\mathfrak{p}$ is flat; to this end observe that we have an inclusion of multiplicative sets $S := R[X] \setminus \mathfrak{q} \subset T := R[X] \setminus \mathfrak{p}$ and by §16, Prop. 8, we have $T^{-1}(R[X]_\mathfrak{q}) = T^{-1}S^{-1}(R[X]) = T^{-1}(R[X])$. Thus $T^{-1}(\varphi)$ is equal to the localization of the natural inclusion $R \hookrightarrow R[X]$; since $R[X]$ is a free R module the latter is flat and hence so is $T^{-1}(\varphi)$ and also φ.] By §23, Thm 8 we get

$$(*) \quad \dim(R[X]_\mathfrak{q}) \leq \dim R_\mathfrak{p} + \dim \left(\frac{R[X]_\mathfrak{q}}{\mathfrak{p} R[X]_\mathfrak{q}} \right).$$

Now set $S := R \setminus \mathfrak{p}$ and $T := R[X] \setminus \mathfrak{q}$. Then S and T are multiplicative subsets of $R[X]$ and by definition of $\mathfrak{p}$ we have $S \subset T$. We have $S^{-1}(R[X]) \cong R_\mathfrak{p}[X]$ (you can check this directly by hand or using the UMP). By §16, Prop 8

$$R[X]_\mathfrak{q} = T^{-1}(R[X]) \cong T^{-1}(S^{-1}(R[X])) \cong T^{-1}(R_\mathfrak{p}[X]).$$

By §16, Cor 6 we get

$$\frac{R[X]_\mathfrak{q}}{\mathfrak{p} R[X]_\mathfrak{q}} \cong T^{-1} \left(\frac{R_\mathfrak{p}[X]}{\mathfrak{p} R_\mathfrak{p}[X]} \right) \cong T^{-1} \left(\frac{R_\mathfrak{p}}{\mathfrak{p} R_\mathfrak{p}}[X] \right).$$

By §16, Cor 10 we have

$$K := \mathrm{Frac}(R/\mathfrak{p}) \cong R_\mathfrak{p}/\mathfrak{p} R_\mathfrak{p}.$$

Thus all together

$$\frac{R[X]_\mathfrak{q}}{\mathfrak{p} R[X]_\mathfrak{q}} = T^{-1}(K[X]) = K[X]_\mathfrak{q}.$$

Plugging this into (*) we get

$$dim(R[X]_{\mathfrak{q}}) \leq \dim R_{\mathfrak{p}} + \dim K[X]_{\mathfrak{q}} \leq \dim R_{\mathfrak{p}} + 1 \leq \dim R + 1.$$

Hence

$$\dim R[X] = \sup_{\mathfrak{q}}(\dim(R[X]_{\mathfrak{q}})) \leq \dim R + 1.$$

This proves the statement. $\square$