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Exercise sheet 9 for Algebra II

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Exercise 1 (Faithfully flat descent)**.** Let $\varphi : A \to B$ be a ring homomorphism, which is faithfully flat (i.e., if we view B as an A-algebra via φ, then B is faithfully flat over A.) Define $d : B \to B \otimes_A B$ by $d(b) = 1 \otimes b - b \otimes 1$; it is an A-linear map. Show that the sequence

$$(*) \quad 0 \to A \xrightarrow{\varphi} B \xrightarrow{d} B \otimes_A B$$

is an exact sequence of A-modules, in particular $A \cong \mathrm{Ker}(d)$. To this end proceed as follows:

(i) Consider the sequence $(*) \otimes_A B$, i.e.,

$$(2*) \quad 0 \to B \xrightarrow{d_0} B \otimes_A B \xrightarrow{d_1} B \otimes_A B \otimes_A B,$$

where d_0 is the composition of the isomorphism $B \cong A \otimes_A B$ with $\varphi \otimes \mathrm{id}_B$ and $d_1 = d \otimes \mathrm{id}_B$. Show that there are well defined A-linear maps $s_0 : B^{\otimes 2} \to B$, with $s_0(b_1 \otimes b_2) = b_1 b_2$, and $s_1 : B^{\otimes 3} \to B^{\otimes 2}$, with $s_1(b_1 \otimes b_2 \otimes b_3) = b_1 \otimes b_2 b_3$.

(ii) Show that $\mathrm{id}_B = s_0 \circ d_0$ and $\mathrm{id}_{B^{\otimes 2}} = d_0 \circ s_0 - s_1 \circ d_1$.

(iii) Conclude that $(2*)$ is exact.

(iv) Conclude that $(*)$ is exact.

Exercise 2. Let L/K be a finite field extension which is Galois, i.e., $L \cong K[x]/(f)$, where $f \in K[x]$ is an irreducible polynomial of degree n such that $f = \prod_i (x - a_i) \in L[x]$, with $a_i \neq a_j$, for $i \neq j$. Set $\mathrm{Gal}(L/K) = \{K\text{-algebra automorphisms of } L\}$. Show

$$L^{\mathrm{Gal}(L/K)} := \{b \in L \,|\, \sigma(b) = b \text{ for all } \sigma \in \mathrm{Gal}(L/K)\} = K.$$

This is one of the main statements of Galois theory. (*Hint:* Show that L/K is faithfully flat. Then use the description of $L \otimes_K L$ from Exercise 2 on sheet 7 and Exercise 1 above.)

Exercise 3. Let R be a ring and $S \subset R$ a multiplicative subset.

(i) Show that $S^{-1}R$ is a flat R-algebra. (*Hint:* Ideal criterion for flatness.)

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(ii) Let M be an R-module. Show that

$$\mathrm{Ker}(M \to S^{-1}R \otimes_R M) = \{m \in M \mid \exists s \in S \text{ such that } sm = 0\}.$$

Exercise 4. Let A be a local domain, i.e. a local ring which is a domain, with maximal ideal $\mathfrak{m}$, residue field $k = A/\mathfrak{m}$ and fraction field $K = \mathrm{Frac}(A)$. Let M be a finitely generated A-module. Assume that we have the equality of vector space dimensions $\dim_K(M \otimes_A K) = \dim_k(M \otimes_A k) =: n$. Show that M is a free A-module of rank n. (*Hint:* Use a Corollary of Nakayama's Lemma to see that M is generated by n elements. This yields an exact sequence $0 \to L \to A^n \to M \to 0$. Use Exercise 3, (i) to conclude $L \otimes_A K = 0$ and Exercise 3, (ii) to conclude $L = 0$.)

Definition 5. *Let R be a domain with fraction field $K = \mathrm{Frac}(R)$. The normalization $\tilde{R}$ of R is the integral closure of R in K. We say that R is* **normal** *if it is integrally closed in K, i.e. if $R = \tilde{R}$.*

Exercise 6. Let R be a domain and $S \subset R$ a multiplicative subset with $0 \notin S$. Show that the normalization of $S^{-1}R$ is the localization at S of the normalization, i.e., $\widetilde{S^{-1}R} = S^{-1}\tilde{R}$. In particular, if R is normal, then so is $S^{-1}R$.

Exercise 7. Show that a UFD is normal.

Exercise 8. Let k be a field and set $R := k[X,Y]/(Y^2 - X^2 - X^3)$ and denote by $x = \bar{X}$ and $y = \bar{Y}$ the images of X and Y in R.

(i) Show that R is a domain.
(ii) Set $t := y/x \in \mathrm{Frac}(R) =: K$. Show that $k[t] \subset K$ is isomorphic as k-algebra to the polynomial ring with one variable and coefficients in k.
(iii) Show that $\mathrm{Frac}(R) = k(t)$. Here $k(t) = \mathrm{Frac}(k[t])$.
(iv) Show that $k[t]$ is the normalization of R. (*Hint:* First observe using 7 that it suffices to show that $k[t]$ contains R and is integral over R.)