06. December 2016

Exercise sheet 8 for Algebra II

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Exercise 1. Prove the equational criterion for flatness: An R-module M is flat if and only if for any relation $\sum_{i=1}^r x_i m_i = 0$, with $r \geq 1$, $x_i \in R$ and $m_i \in M$, $i = 1, \ldots, r$, there exist an $s \geq 1$ and elements $m'_j \in M$, $j = 1, \ldots, s$, $x_{i,j} \in R$, $i = 1, \ldots, r$, $j = 1, \ldots, s$ such that $m_i = \sum_{j=1}^s x_{i,j} m'_j$, for all i, and $\sum_{i=1}^r x_{i,j} x_i = 0$, for all j. (*Hint*: Use the equivalence (i)$\Leftrightarrow$ (iv) of §11, Theorem 9.)

Exercise 2. Let R be a ring and M an R-module. Recall that for $x \in R$ we have $\mathrm{Ann}(x) = \{y \in R \mid yx = 0\}$.

- (i) Show that if M is flat, then $xm = 0$, for $x \in R$ and $m \in M$, implies $m \in \mathrm{Ann}(x) \cdot M$.
- (ii) Assume that every ideal in R is principal (but it does not need to be a domain). Show that, if $xm = 0$, for $x \in R$ and $m \in M$, implies $m \in \mathrm{Ann}(x) \cdot M$, then M is flat.
- (iii) Assume R is a principal ideal domain. Show that M is flat if and only if it is torsion free, i.e., $xm = 0$, $x \in R$, $m \in M$, implies $x = 0$ or $m = 0$.

Exercise 3. Let R be a ring and $I \subset R$ an ideal.

- (i) Show that if R/I is a flat R-module, then $I^2 = I$. (*Hint*: First show that $I \otimes_R R/I \cong I/I^2$.)
- (ii) Show that if I is finitely generated and $I = I^2$, then R/I is a projective R-module, in particular it is flat. (*Hint*: First use a corollary of the Cayley-Hamilton Theorem (determinant trick), to show that I is principal and hence is generated by an idempotent element.)

Exercise 4. We saw in the lecture that if $\varphi : M \to M$ is an R-linear and surjective endomorphism of a *finitely generated* R-module M, then φ is an isomorphism. Show by an example that this does not need to be the case if M is not finitely generated. (*Hint*: You can for example consider the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ with φ the multiplication-by-n-map.)

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