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# Exercise sheet 7 for Algebra II

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**Exercise 1.** Let $R$ be a ring and $I, J \subset R$ two ideals. Show that $R/I \otimes_R R/J \cong R/(I+J)$.

**Exercise 2.** Let $K$ be a field and $f \in K[X]$ an irreducible polynomial. Let $L/K$ be a field extension which contains all roots of $f$, i.e., $f = \prod_{i=1}^n (X - a_i)$ with $a_i \in L$. Assume that $f$ is separable, i.e., $a_i \neq a_j$, for all $i \neq j$. Set $K_f := K[X]/(f)$, from Exercise sheet 2 we know that $K_f$ is a field. Denote by $\alpha \in K_f$ the class of $X$ mod $(f)$.

(i) Show that there are unique $K$-algebra homomorphisms $\sigma_i : K_f \to L$ such that $\sigma_i(\alpha) = a_i$, $i = 1, \ldots, n$.

(ii) Show that there is a unique isomorphism of $K$-algebras $K_f \otimes_K L \xrightarrow{\simeq} \prod_{i=1}^n L$ sending $b \otimes \lambda$ to $(\sigma_1(b)\lambda, \ldots, \sigma_n(b)\lambda)$.

**Exercise 3.** Let $\varphi : R \to R'$ be a ring homomorphism. Recall we that have the functor $\varphi_* : (R'\text{-mod}) \to (R\text{-mod})$ which sends an $R'$-module $N$ to the $R$-module with the same underlying group and scalar multiplication induced via $\varphi$. In the following we view $R'$ as an $R$-module via $\varphi$.

(i) Let $M$ be an $R$-module. Show that $M \otimes_R R'$ is naturally an $R'$-module.

(ii) Show that we get a functor $(R\text{-mod}) \to (R'\text{-mod})$, $M \mapsto \varphi^*(M) := M \otimes_R R'$.

(iii) Show that there is a natural transformation $\eta : \mathrm{id}_{(R\text{-mod})} \to \varphi_* \circ \varphi^*$ such that $\eta(M) : M \to \varphi_*\varphi^* M = M \otimes_R R'$ is given by $m \mapsto m \otimes 1$.

(iv) Show that $\varphi^*$ is left adjoint to $\varphi_*$. (*Hint:* Use Exercise 5.2, (b).)

(v) Show that for $R$-modules $M$ and $M'$ and an $R'$-module $N$ we have the following formulas:

$$\varphi_*(\varphi^* M \otimes_{R'} N) \cong M \otimes_R \varphi_* N, \quad \varphi^*(M \otimes_R M') \cong \varphi^* M \otimes_{R'} \varphi^*(M').$$

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**Exercise 4.** Let $R$ and $R'$ be two rings and $F : (R\text{-mod}) \to (R'\text{-mod})$ be a functor. Show that the following statements are equivalent:

(i) $F$ is exact.

(ii) If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $R$-modules, then $0 \to F(M') \to F(M) \to F(M'') \to 0$ is a short exact sequence of $R'$-modules.

(iii) $F$ is left exact, and, if $M \to M''$ is surjective, then so is $F(M) \to F(M'')$.

(iv) $F$ is right exact, and, if $M' \to M$ is injective, then so is $F(M') \to F(M)$.

(v) If $\alpha : M \to N$ is an $R$-linear map, then $F(\text{Ker}(\alpha)) = \text{Ker}(F(\alpha))$ and $F(\text{Im}(\alpha)) = \text{Im}(F(\alpha))$.