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Exercise sheet 6 for Algebra II

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Exercise 1. Let R be a ring. Show that any R-module is a filtered direct limit over its finitely generated submodules. (*Hint:* First show that the set of all finitely generated submodules of a given R-module M form a directed partially ordered set, with the partial order given by the inclusion. Then go to the limit!)

Exercise 2. Let R be a ring, I a filtered category and $M : I \to (R\text{-mod})$ a functor. Denote by $\alpha_i^M : M_i \to \varinjlim_I M$ the natural R-linear maps.

(i) Show that for R-modules N there is a natural R-linear map

$$\theta(N) : \varinjlim_I(\mathrm{Hom}_R(N, M)) \to \mathrm{Hom}_R(N, \varinjlim_I M),$$

which is unique with the property that we have the following equality

$$\theta(N)\circ\alpha_i^{\mathrm{Hom}_R(N,M)} = \mathrm{Hom}_R(N, \alpha_i^M) : \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, \varinjlim_I M).$$

(ii) Show that if $\varphi : N' \to N$ is an R-linear map then we have the following equality of maps between $\varinjlim_I(\mathrm{Hom}_R(N, M)) \to \mathrm{Hom}_R(N', \varinjlim_I M)$

$$\theta(N') \circ \varinjlim_I(\mathrm{Hom}_R(\varphi, M)) = \mathrm{Hom}_R(\varphi, \varinjlim_I M) \circ \theta(N).$$

(iii) Show that for all natural numbers $n \geq 0$ the R-linear map $\theta(R^n) : \varinjlim_I(\mathrm{Hom}_R(R^n, M)) \xrightarrow{\simeq} \mathrm{Hom}_R(R^n, \varinjlim_I M)$ is an isomorphism. (*Hint:* Use the compatibility of $\mathrm{Hom}_R(-, -)$ and $\varinjlim_I$ with direct sums proved in the lecture.)

(iv) Show that $\theta(N) : \varinjlim_I(\mathrm{Hom}_R(N, M)) \to \mathrm{Hom}_R(N, \varinjlim_I M)$ is injective if N is finitely generated and bijective if N is finitely presented. (*Hint:* If N is finitely generated we find a free presentation $R^{\oplus\Sigma} \to R^n \to N \to 0$, with Σ a finite set in case

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N is finitely presented. Then use the exactness properties of Hom_R, $\varinjlim$, the Snake Lemma and the above.)

Exercise 3. Let A be an R-algebra.

(i) Show that there is a well defined R-linear map $\mu : A \otimes_R A \to A$ such that $\mu(a \otimes b) = ab$.
(ii) Show that $\mu : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ is an isomorphism.
(iii) Show that $\mu : R[x] \otimes_R R[x] \to R[x]$ is surjective but not injective.

Exercise 4. Let M and N be free R-modules with respective bases $\{m_\lambda\}_{\lambda \in \Lambda}$ and $\{n_\sigma\}_{\sigma \in \Sigma}$. Show that $M \otimes_R N$ is free with basis $\{m_\lambda \otimes n_\sigma\}_{(\lambda,\sigma) \in \Lambda \times \Sigma}$. In particular, $R^m \otimes_R R^n \cong R^{mn}$.