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Exercise sheet 5 for Algebra II

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Exercise 1. Let R be a ring. Denote by $A = R[x_i \,|\, i \in \mathbb{N}]$ the polynomial ring in countably many variables and by $I =< x_i, i \in \mathbb{N} >\subset A$ the ideal generated by all the variables. Set $M := A/I$; it has a natural A-module structure (via the quotient map) and via the inclusion $R \hookrightarrow A$ also an R-module structure.

(i) Is M finitely presented as A-module?
(ii) Is M finitely presented as R-module?

Exercise 2. Let $F : C \to D$ and $G : D \to C$ be two functors between categories.

(i) Assume that F is left adjoint to G, i.e. we have a bijection $\varphi_{X,Y} : \mathrm{Hom}_D(F(X), Y) \xrightarrow{\simeq} \mathrm{Hom}_C(X, G(Y))$, for all $X \in C$ and $Y \in D$, which is natural in the sense discussed in the lecture.
  (a) Show that there is a natural transformation of functors $\eta : \mathrm{id}_C \to G \circ F$, such that $\eta(X) = \varphi_{X,F(X)}(\mathrm{id}_{F(X)}) : X \to G(F(X))$.
  (b) Show that for any morphism $g : X \to G(Y)$ in C there is a unique morphism $f : F(X) \to Y$ in D such that $g = G(f) \circ \eta(X)$.

(ii) Assume there is a natural transformation $\eta : \mathrm{id}_C \to G \circ F$ as in (a) satisfying (b). Show that in this case F is left adjoint to G.

Exercise 3. Let R be a ring and $I \subset R$ an ideal. Denote by $\pi : R \to R/I$ the quotient map. Recall that if M is an R/I-module then we denote by $\pi_* M$ the R-module whose underlying abelian group is the one from M and scalar multiplication is defined by $x \cdot m := \pi(x)m$, $x \in R$, $m \in M$. Show that we obtain a functor $\pi_* : (R/I-\mathrm{mod}) \to (R-\mathrm{mod})$ and that it has a left adjoint, which is on objects given by

$$\pi^* : (R - \mathrm{mod}) \to (R/I - \mathrm{mod}), \quad N \mapsto \pi^*(N) := N/IN.$$

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