Exercise sheet 5 for Algebra II

Kay Rülling

Exercise 1. Let $R$ be a ring. Denote by $A = R[x_i \mid i \in \mathbb{N}]$ the polynomial ring in countably many variables and by $I = \langle x_i, i \in \mathbb{N} \rangle \subset A$ the ideal generated by all the variables. Set $M := A/I$; it has a natural $A$-module structure (via the quotient map) and via the inclusion $R \hookrightarrow A$ also an $R$-module structure.

(i) Is $M$ finitely presented as $A$-module?
(ii) Is $M$ finitely presented as $R$-module?

Exercise 2. Let $F : C \to D$ and $G : D \to C$ be two functors between categories.

(i) Assume that $F$ is left adjoint to $G$, i.e. we have a bijection

$$\varphi_{X,Y} : \text{Hom}_D(F(X), Y) \cong \text{Hom}_C(X, G(Y)),$$

for all $X \in C$ and $Y \in D$, which is natural in the sense discussed in the lecture.

(a) Show that there is a natural transformation of functors

$$\eta : \text{id}_C \to G \circ F,$$

such that $\eta(X) = \varphi_{X,F(X)}(\text{id}_{F(X)}) : X \to G(F(X))$.

(b) Show that for any morphism $g : X \to G(Y)$ in $C$ there is a unique morphism $f : F(X) \to Y$ in $D$ such that $g = G(f) \circ \eta(X)$.

(ii) Assume there is a natural transformation $\eta : \text{id}_C \to G \circ F$ as in (a) satisfying (b). Show that in this case $F$ is left adjoint to $G$.

Exercise 3. Let $R$ be a ring and $I \subset R$ an ideal. Denote by $\pi : R \to R/I$ the quotient map. Recall that if $M$ is an $R/I$-module then we denote by $\pi_* M$ the $R$-module whose underlying abelian group is the one from $M$ and scalar multiplication is defined by $x \cdot m := \pi(x)m$, $x \in R$, $m \in M$. Show that we obtain a functor $\pi_* : (R/I - \text{mod}) \to (R - \text{mod})$ and that it has a left adjoint, which is on objects given by

$$\pi^* : (R - \text{mod}) \to (R/I - \text{mod}), \quad N \mapsto \pi^*(N) := N/IN.$$

1Questions or comments to kay.ruelling@fu-berlin.de or come to 1.103(RUD25) on Tue/Thu/Fri.