08. November 2016

Exercise sheet 4 for Algebra II

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Exercise 1. Let R be a ring and $I \subset R$ an ideal. Show that I is free as an R-module if and only if $I = (x)$, where $x \in R$ is a nonzerodivisor.

Exercise 2. Let R be a ring and $A = R[x, y]$ the polynomial ring in two variables with coefficients in R. Give free finite presentations of the ideals (x, y) and (x^2, xy, y^2), viewed as A-modules.

Exercise 3. Let $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ be an exact sequence of R modules. Assume α is injective and β has a section, i.e. there is an R-linear map $\sigma : M'' \to M$ such that $\beta \circ \sigma = \mathrm{id}_{M''}$. Show that then β is surjective, α has a retraction and the sequence splits.

Exercise 4. Let $0 \to N' \xrightarrow{a} N \xrightarrow{b} N''$ be a sequence of R-modules. Show that it is exact if and only if for all R-modules M the sequence $0 \to \mathrm{Hom}_R(M, N') \xrightarrow{a_*} \mathrm{Hom}_R(M, N) \xrightarrow{b_*} \mathrm{Hom}_R(M, N'')$ is exact. (Here $a_*(\varphi) = a \circ \varphi$ and similar with b_*.)

Exercise 5. Let R be a ring, $x \in R$ a nonzerodivisor and $r, s \geq 1$ positive integers. Show that there is a well-defined R-linear map $\underline{x}^r : R/(x^s) \to R/(x^{s+r})$, $a \bmod (x^s) \mapsto x^r a \bmod (x^{r+s})$, which fits in a short exact sequence of R-modules

$$0 \to R/(x^s) \xrightarrow{\underline{x}^r} R/(x^{r+s}) \to R/(x^r) \to 0.$$

Can this sequence be split?