Exercise 1. (i) Show that $\mathbb{Z}[i] := \mathbb{Z}[x]/<x^2+1>$ and $\mathbb{Z}[\sqrt{-5}] := \mathbb{Z}[x]/<x^2+5>$ are domains (Hint: Use Exercise sheet 2.)
(ii) Show that $\mathbb{Z}[i]$ is an euclidean domain. In particular, it is a UFD.
(iii) Show that if $p \in \mathbb{Z}$ is a prime number, then $p \cdot \mathbb{Z}[i]$ is a prime ideal if and only if $-1$ is not a square in $\mathbb{Z}/<p>$.
(iv) Show that $2 \in \mathbb{Z}[\sqrt{-5}]$ is irreducible but not prime. Hence $\mathbb{Z}[\sqrt{-5}]$ is not a UFD. (Hint: To show that 2 is not prime try to factor $6 \in \mathbb{Z}[\sqrt{-5}]$ in two different ways.)

Exercise 2. Let $R$ be an integral domain and $R[X,Y]$ the polynomial ring in two variables with coefficients in $R$. Let $m,n \in \mathbb{Z}_{\geq 1}$ be positive integers.

(i) Show that the ideal $<X^m - Y^n>$ is prime in $R[X,Y]$ if and only if $m$ and $n$ are coprime, i.e. $<m,n> = \mathbb{Z}$.
(Hint: For the "if" direction: Show that the map $\varphi : R[X,Y] \to R[T], f(X,Y) \mapsto f(T^n,T^m)$ is a ring homomorphism, which factors over a ring homomorphism $\bar{\varphi} : R[X,Y]/<X^m - Y^n> \to R[T]$. Then show that $\bar{\varphi}$ is injective.)
(ii) Let $K$ be a field. Show that $y = \bar{Y} \in K[X,Y]/<Y^2 - X^3>$ is irreducible but not prime.

Exercise 3. For a ring $R$ we denote by $\text{Spec } R$ its set of prime ideals. Let $\varphi : R \to R'$ be a ring homomorphism. From the lecture we know that this induces a map (of sets) $\varphi^{-1} : \text{Spec } R' \to \text{Spec } R, p \mapsto f^{-1}(p)$.

(i) Let $R_1, \ldots, R_n$ be rings, denote by $R = R_1 \times \ldots \times R_n$ their product and by $\pi_i : R \to R_i, (a_1, \ldots, a_n) \mapsto a_i, i = 1 \ldots, n$, the projection maps.
Show that $\pi_i^{-1} : \text{Spec } R_i \to \text{Spec } R$ maps bijectively onto $\pi_i^{-1}(\text{Spec } R_i)$ and that we have the following decomposition of $\text{Spec } R$ into disjoint sets
$\text{Spec } R = \pi_1^{-1}(\text{Spec } R_1) \sqcup \ldots \sqcup \pi_n^{-1}(\text{Spec } R_n) \bij \text{Spec } R_1 \sqcup \ldots \sqcup \text{Spec } R_n$.
(ii) Let $\pi : R \to R_{\text{red}} := R/\text{nil}(R)$ be the canonical surjection. Show that $\pi^{-1} : \text{Spec } R_{\text{red}} \to \text{Spec } R$ is bijective.
Exercise 4. Let $R$ be a reduced ring with minimal prime ideals $p_\lambda$, $\lambda \in \Lambda$. Show that the ring homomorphism $R \to \prod_{\lambda \in \Lambda} R/p_\lambda$, $x \mapsto (x \mod p_\lambda)$ is injective.