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Exercise sheet 2 for Algebra II

Kay Rülling

2.1. We recall the following definitions (e.g. from Algebra I): Let R be a domain.

- R is an *euclidean domain* if there exists a function $\varphi : R \to \mathbb{N}_0 \cup \{-\infty\}$ with $\varphi(a) = -\infty \Leftrightarrow a = 0$, and such that for all $a, b \in R$, $b \neq 0$, there exist $q, r \in R$ with
$$a = qb + r, \quad \varphi(r) < \varphi(b).$$
- R is a *principal ideal domain* (PID) if every ideal in R is principal.
- R is a *unique factorization domain (UFD)* if every element $a \in R$ which is neither zero nor a unit can be written as a product of prime elements $a = p_1 \cdots p_n$. (An element $p \in R$ is *prime* iff the ideal $(p) \subset R$ is a prime ideal.)

Exercise 1.
(i) Show that an euclidean domain is a PID.
(ii) Show that $\mathbb{Z}$ and $K[x]$, with K a field, are euclidean domains.

Exercise 2.
(i) Let R be a PID and $\mathfrak{p} \subset R$ a prime ideal. Then either $\mathfrak{p} =< 0 >$ or there exists a prime element $p \in R$ such that $\mathfrak{p} =< p >$.
(ii) Any non-zero prime ideal in a PID is maximal.
(iii) Show that a PID is a UFD. (*Hint:* First show that in a PID any ascending chain of ideals $I_1 \subset I_2 \subset \ldots$ becomes stationary, i.e. we have $I_n = I_{n+1}$ for all n large enough. Then show that in a PID any non-zero element a which is not a unit can be written as $a = pa_1$ with p a prime. Continue with a_1 and so on and deduce the statement.)

Exercise 3. Let R be a domain.
(i) Assume we can write $a \in R$ as a product of prime elements $a = p_1 \cdots p_n$. Then this presentation is unique up to permutation and multiplication with units, i.e. if $a = p'_1 \cdots p'_m$ with prime elements $p'_i \in R$, then $n = m$ and there exists a bijection $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ and units $u_i \in R^\times$ such that $p'_i = u_i p_{\sigma(i)}$.

(ii) Let R be a domain. Recall that an element $a \in R \setminus \{0\}$ is *irreducible* if it is not a unit and if we can write $a = bc$, then either $b \in R^\times$ or $c \in R^\times$.

Show that a prime element in R is always irreducible. Show that if R is a UFD, then an irreducible element is also prime.

(iii) (Gauss Lemma) We say a polynomial $f = \sum_{i=0}^n a_i X^i \in R[X]$ is *primitive* if the coefficients $a_1, \ldots, a_n$ are not divisible by a common prime element. Let R be a UFD.

Show that if $f, g \in R[X]$ are primitive then so is fg.

Exercise 4. Let R be a domain.

(i) We say $(a, b), (a', b') \in R \times R \setminus \{0\}$ are equivalent if $ab' = a'b$. Denote by K the set of equivalence classes of the pairs (a, b). We write $\frac{a}{b} \in K$ for the equivalence class of (a, b). Show that the operations

$$\frac{a}{b} + \frac{a'}{b'} := \frac{ab' + ba'}{bb'}, \quad \frac{a}{b} \cdot \frac{a'}{b'} := \frac{aa'}{bb'}, \quad -\frac{a}{b} := \frac{-a}{b}$$

are well-defined and give K the structure of a field with neutral elements $0_K = \frac{0}{1}$ and $1_K = \frac{1}{1}$. Further show that $R \to K$, $a \mapsto \frac{a}{1}$ is an injective ring homomorphism. (In the following we will view $R \subset K$ as a subring and identify the elements a and $\frac{a}{1}$.)

(ii) Show that any $f \in R[X] \setminus \{0\}$ can be written as $f = af_0$ where $f_0 \in R[X]$ is primitive (in the sense of Ex.1, (iii)) and $a \in R \setminus \{0\}$.

(iii) Show that if $p \in R$ is a prime element in R, then it is also a prime element in $R[X]$.

(iv) Assume that R is a UFD. Show that if $f \in R[X]$ is primitive and its image in $K[X]$ is prime, then $f \in R[X]$ is also prime. (*Hint:* Use (iii) of Ex. 3 and (ii) above.)

(v) Deduce from (ii), (iii) and (iv) above that if R is a UFD, then so is $R[X]$. (*Hint:* Notice that we know that $K[X]$ is a PID and hence by Ex. 2, (iii) also a UFD.)

Remark 1. Ex. 1, (iv) and Ex. 2(v) together imply that $\mathbb{Z}[X_1, \ldots, X_n]$ and $K[X_1, \ldots, X_n]$ (K a field) are UFD's.