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Exercise sheet 14 for Algebra II

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Exercise 1. Let k be an algebraically closed field. We say a subset $X \subset k^n$ is *algebraic* if there are polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $X = \{a \in k^n \mid f_i(a) = 0,\, i = 1, \ldots, r\}$. Show that there is an inclusion inversing bijection of sets

$$\{\text{algebraic subsets in } k^n\} \xrightarrow{1:1} \{I \subset k[x_1, \ldots, x_n] \text{ ideal, wtih } \sqrt{I} = I\}$$

given by

$$X \mapsto I(X) := \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0, \text{ for all } a \in X\}$$

and with inverse given by

$$Z(I) := \{a \in k^n \mid f(a) = 0, \text{ for all } f \in I\} \leftarrow\!\shortmid I.$$

(*Hint:* Essentially you have to show $I(Z(I)) = I$. To this end first use Hilbert's Nullstellensatz to show that if $\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$ is the maxiaml ideal corresponding to $a = (a_1, \ldots, a_n) \in k^n$, then $\sqrt{I} = \bigcap_{a \in Z(I)} \mathfrak{m}_a$.)

Exercise 2. Let R be a domain with fraction field $K = \mathrm{Frac}(R)$. Then we say that R is a *discrete valuation ring* (DVR) if there exists a surjective function -called discrete valuation- $v : K^\times \to \mathbb{Z}$ satisfying i) $v(ab) = v(a) + v(b)$ and ii) $v(a + b) \geq \min(v(a), v(b))$, such that $R = \{a \in K^\times \mid v(a) \geq 0\} \cup \{0\}$. Show that the following statements for a domain R are equivalent:

(i) R is local ring and a PID.

(ii) R is a DVR.

(*Hint:* For (i)⇒(ii) use the Krull Intersection Theorem to show that if π is a generator of the maximal ideal of R, then any element $x \in R \setminus \{0\}$ can be written uniquely written as $x = \pi^n u$, where $n \in \mathbb{N}_0$ and $u \in R^\times$; then show that there is a unique discrete valuation v on $\mathrm{Frac}(K)^\times$ satisfying $v(x) = n$. For (ii)⇒(i) first show that $R^\times = \{a \in K^\times \mid v(a) = 0\}$; deduce that if $\pi \in R$ satisfies $v(\pi) = 1$, then any element in R can be written as $\pi^n u$ with $n \in \mathbb{N}_0$ and $u \in R^\times$; conclude.)

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Exercise 3. Let R be a ring.

(i) Let $x \in R$ be an element which is neither a unit nor a zerodivisor. Show that for all $n \geq 1$ we have $\mathrm{Ass}_R(R/x^nR) = \mathrm{Ass}_R(R/xR)$. (*Hint:* Consider the short exact sequence $0 \to R/xR \xrightarrow{\cdot x^{n-1}} R/x^nR \to R/x^{n-1}R \to 0$.)

(ii) Let $\mathfrak{p} \subset R$ be a prime ideal. Show $\mathrm{Ass}_R(R/\mathfrak{p}) = \{\mathfrak{p}\}$.

(iii) Let $I \subset R$ be an ideal and denote by $\pi : R \to R/I$ the quotient map. Let M be an R/I-module. Show that there is a bijection $\mathrm{Ass}_{R/I}(M) \to \mathrm{Ass}_R(M)$, $\mathfrak{p} \mapsto \pi^{-1}(\mathfrak{p})$.

Exercise 4. Set $R = k[x,y]/(x^2, xy)$ Compute $\mathrm{Ass}_R(R)$. Which of the primes are minimal, which are embedded? (*Hint*: Compute $\mathrm{Ass}_{k[x,y]}(R)$ and use Exercise 3, (iii). To this end try to use the behavior of Ass under short exact sequences of modules.)