Exercise sheet 14 for Algebra II

Kay Rülling

Exercise 1. Let $k$ be an algebraically closed field. We say a subset $X \subset k^n$ is algebraic if there are polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $X = \{a \in k^n \mid f_i(a) = 0, \ i = 1, \ldots, r\}$. Show that there is an inclusion inversing bijection of sets
\[
\{\text{algebraic subsets in } k^n\} \overset{1:1}{\longrightarrow} \{I \subset k[x_1, \ldots, x_n] \text{ ideal, with } \sqrt{I} = I\}
\]
given by
\[
X \mapsto I(X) := \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0, \text{ for all } a \in X\}
\]
and with inverse given by
\[
Z(I) := \{a \in k^n \mid f(a) = 0, \text{ for all } f \in I\} \mapsto I.
\]
(Hint: Essentially you have to show $I(Z(I)) = I$. To this end first use Hilbert’s Nullstellensatz to show that if $m_a = (x_1 - a_1, \ldots, x_n - a_n)$ is the maximal ideal corresponding to $a = (a_1, \ldots, a_n) \in k^n$, then $\sqrt{I} = \bigcap_{a \in Z(I)} m_a$.)

Exercise 2. Let $R$ be a domain with fraction field $K = \text{Frac}(R)$. Then we say that $R$ is a discrete valuation ring (DVR) if there exists a surjective function -called discrete valuation- $v : K^\times \rightarrow \mathbb{Z}$ satisfying
i) $v(ab) = v(a) + v(b)$ and ii) $v(a + b) \geq \min(v(a), v(b))$, such that $R = \{a \in K^\times \mid v(a) \geq 0\} \cup \{0\}$. Show that the following statements for a domain $R$ are equivalent:

(i) $R$ is local ring and a PID.
(ii) $R$ is a DVR.

(Hint: For (i)⇒(ii) use the Krull Intersection Theorem to show that if $\pi$ is a generator of the maximal ideal of $R$, then any element $x \in R \setminus \{0\}$ can be written uniquely written as $x = \pi^n u$, where $n \in \mathbb{N}_0$ and $u \in R^\times$; then show that there is a unique discrete valuation $v$ on $\text{Frac}(K)^\times$ satisfying $v(x) = n$. For (ii)⇒(i) first show that $R^\times = \{a \in K^\times \mid v(a) = 0\}$; deduce that if $\pi \in R$ satisfies $v(\pi) = 1$, then any element in $R$ can be written as $\pi^n u$ with $n \in \mathbb{N}_0$ and $u \in R^\times$; conclude.)

Questions or comments to kay.ruelling@fu-berlin.de or come to 1.103(RUD25) onTue/Thu/Fri.
**Exercise 3.** Let $R$ be a ring.

(i) Let $x \in R$ be an element which is neither a unit nor a zero divisor. Show that for all $n \geq 1$ we have $\text{Ass}_R(R/x^nR) = \text{Ass}_R(R/xR)$. (*Hint:* Consider the short exact sequence $0 \to R/xR \xrightarrow{x^{n-1}} R/x^nR \to R/x^{n-1}R \to 0$.)

(ii) Let $p \subset R$ be a prime ideal. Show $\text{Ass}_R(R/p) = \{p\}$.

(iii) Let $I \subset R$ be an ideal and denote by $\pi : R \to R/I$ the quotient map. Let $M$ be an $R/I$-module. Show that there is a bijection $\text{Ass}_{R/I}(M) \to \text{Ass}_R(M), p \mapsto \pi^{-1}(p)$.

**Exercise 4.** Set $R = k[x, y]/(x^2, xy)$ Compute $\text{Ass}_R(R)$. Which of the primes are minimal, which are embedded? (*Hint:* Compute $\text{Ass}_{k[x, y]}(R)$ and use Exercise 3, (iii). To this end try to use the behavior of $\text{Ass}$ under short exact sequences of modules.)