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Exercise sheet 13 for Algebra II

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Exercise 1. Let k be a field and $k[X, Y, Z]$ the polynomial ring in three variables. Set $R := k[X, Y, Z]/(X^3YZ + X^2Y^3 + YZ + X + Y + 1)$. Find algebraically independent elements $t_2, t_3 \in R$ such that R is finite over $P := k[t_2, t_3]$ the smallest k-subalgebra of R containing t_2, t_3. (*Hint:* Apply the method in the proof of the Noether Normalization; you can choose t_2, t_3 as linear combinations of X, Y, Z.)

Exercise 2. Let R be a ring. Show: $\dim R = \max_{\mathfrak{p}_0}\{\dim(R/\mathfrak{p}_0)\}$, where $\mathfrak{p}_0$ runs through the set of minimal prime ideals.

Exercise 3. In each of the following cases say what are the minimal prime ideals of the ring R and what is its dimension:

(i) $R = k[X_1, \ldots, X_n]/(f)$, where k is a field, $X_1, \ldots, X_n$ are variables, $f \in k[X_1, \ldots, X_n]$ is a non-constant polynomial.
(ii) $R = \mathbb{Z}/p^2\mathbb{Z}$, where $p \in \mathbb{Z}$ is a prime number.
(iii) $R = \mathbb{Z}[X]/(6X^2)$.

Exercise 4. Let $R = \mathbb{Z}_{(p)}[X]$, where p is a prime number and X is a variable. Show that the ideals $(Xp - 1) \subset R$ and $(X, p) \subset R$ are maximal and that the chains $(0) \subset (Xp - 1)$ and $(0) \subset (p) \subset (p, X)$ are maximal.

Exercise 5. A ring R is called *catenary* if for any inclusion of prime ideals $\mathfrak{p} \subset \mathfrak{q}$ in R, all the maximal chains of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r = \mathfrak{q}$ are finite and have the same length.

Show that a finitely generated k-algebra is catenary.

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