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Exercise sheet 12 for Algebra II

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Exercise 1. Let $n \in \mathbb{Z}$ be a square-free integer (i.e. $n = \pm p_1 \cdots p_r$, with prime numbers p_i and $p_i \neq p_j$, for $i \neq j$). Denote by $\mathbb{Z}[\sqrt{n}]$ the smallest subring of the complex numbers $\mathbb{C}$, which contains $\mathbb{Z}$ and $\sqrt{n}$, where $\sqrt{n}$ denotes a root in $\mathbb{C}$ of $X^2 - n$. Denote by $K := \mathbb{Q}(\sqrt{n})$ the fraction field of $\mathbb{Z}[\sqrt{n}]$ and by R the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{n})$. Show

$$R = \begin{cases} \mathbb{Z}[\sqrt{n}] & \text{if } n \not\equiv 1 \bmod 4\mathbb{Z} \\ \mathbb{Z}[\frac{1}{2} + \frac{1}{2}\sqrt{n}] & \text{if } n \equiv 1 \bmod 4\mathbb{Z}. \end{cases}$$

Proceed as follows:

(i) Show this '$\supset$' inclusion.
(ii) Show any element in K can be written in the form $\alpha = a + b\sqrt{n}$, with $a, b \in \mathbb{Q}$.
(iii) Show that the minimal polynomial of $\alpha = a + b\sqrt{n}$, with $b \neq 0$, is $X^2 - 2aX + a^2 - b^2 n \in \mathbb{Q}[X]$.
(iv) Conclude $\alpha \in R \Longleftrightarrow 2a \in \mathbb{Z}$ and $a^2 - b^2 n \in \mathbb{Z}$.
(v) Show that if $\alpha = a + b\sqrt{n} \in R$, then $a, b \in \frac{1}{2}\mathbb{Z}$. (*Hint:* Use n is square-free.)
(vi) Show that if $\alpha = a + b\sqrt{n} \in R$, then there exists an $\alpha_0 \in \{0, \frac{1}{2}, \frac{1}{2}\sqrt{n}, \frac{1}{2} + \frac{1}{2}\sqrt{n}\}$ and an $\alpha_1 \in \mathbb{Z}[\sqrt{n}]$ such that $\alpha = \alpha_0 + \alpha_1$.
(vii) Show that $\frac{1}{2}$ and $\frac{1}{2}\sqrt{n}$ are never integral over $\mathbb{Z}$ and $\frac{1}{2} + \frac{1}{2}\sqrt{n}$ is integral over $\mathbb{Z}$ if and only if $n \equiv 1 \bmod 4$.
(viii) Conclude.

Exercise 2. (i) Show that the integral closure R of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{5})$ is isomorphic to $\mathbb{Z}[X]/(X^2 - X - 1)$. (*Hint:* Use Exercise 1).
(ii) Give all the prime ideals in R lying over the prime ideals (2), (3) and (5) of $\mathbb{Z}$, respectively. (*Hint:* First note that here a prime ideal $\mathfrak{q} \subset R$ lies over $(p) \subset \mathbb{Z}$ iff $\mathfrak{q} \supset pR$. Then investigate R/pR.)

Exercise 3. Let $R_1, \ldots, R_n$ be rings and set $R := \prod_{i=1}^n R_i$. Show that $\dim R = \max_{i=1}^n \{\dim R_i\}$. (*Hint*: Use Exercise 3.3, (i) .)

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Exercise 4. Let k be a field and $k[x, y, z]$ the polynomial ring in three variables with coefficients in k. Denote by $R := k[x, y, z]/(zx, zy)$.

(i) What are the minimal prime ideals of R?
(ii) Show $\dim R = \dim k[x, y] \geq 2$.