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Exercise sheet 11 for Algebra II

Kay Rülling[1]

Exercise 1. Let R be a domain with fraction field K. A *fractional ideal* of R is by definition a finitely generated R-submodule of K.

(i) Show that any fractional ideal J of R is isomorphic to an ideal. (*Hint:* Multiply J by a suitable element of R.)

Recall that an R-module M is torsion-free if $M \to M \otimes_R K$, $m \mapsto m \otimes 1$, is injective. If M is additionally finitely generated, then we say M has rank n if the K-vector space $M \otimes_R K$ has dimension n.

(ii) Show that if M is a finitely generated torsion-free R-module of rank n, then there exist elements $m_1, \ldots, m_n \in M$ such that $m_1 \otimes 1, \ldots, m_n \otimes 1$ is a K-basis of $M \otimes_R K$.

(iii) Let R be a PID and M be a finitely generated torsion-free R-module of rank n. Show that there exists an exact sequence of finitely generated torsion-free R-modules $0 \to N \to M \to M/N \to 0$, where $\mathrm{rank}(N) = n - 1$ and $\mathrm{rank}(M/N) = 1$.

(iv) Show that any finitely generated torsion free R-module of rank 1 is isomorphic to a fractional ideal.

(v) Assume R is a PID and M is a finitely generated R-module. Show : M is a free R-module $\Leftrightarrow$ M is a torsion-free R-module. (*Hint:* "$\Rightarrow$" is clear, to prove "$\Leftarrow$" proceed as follows: Use (iv) and (i) to show that if M has rank 1, then it is free of rank 1 and in particular projective. Then write an exact sequence as in (iii), observe that it splits and conclude by induction.)

Exercise 2. Show that $\mathbb{Q}$ is flat as a $\mathbb{Z}$-module but not projective.

Exercise 3. Show that an R-module P is locally free of rank n if and only if $P_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$-module of rank n, for all maximal ideals $\mathfrak{m} \subset R$.

Exercise 4. Exercise 9.7 and Exercise 10.3.

[1]Questions or comments to kay.ruelling@fu-berlin.de or come to 1.103(RUD25) on Tue/Thu/Fri.