January 3, 2017

Exercise sheet 10 for Algebra II

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Exercise 1. Let R be a ring, $S \subset R$ a multiplicative subset and R' an R-algebra with structure map $\varphi : R \to R'$. Set $T := \varphi(S)$.

(i) Show that $T \subset R'$ is a multiplicative subset.
(ii) Show that the ring $T^{-1}R'$ has a natural structure as $S^{-1}R$-algebra.
(iii) By the above we can consider $T^{-1}R'$ as $S^{-1}R$-module; we can also consider R' as R-module and form the $S^{-1}R$-module $S^{-1}R'$. Show that we have an isomorphism of $S^{-1}R$-modules $T^{-1}R' \cong S^{-1}R'$.

Exercise 2. Let k be a field, $k[x, y]$ the polynomial ring in two variables with coefficients in k and $f \in k[x, y]$. Assume that $f = ax + by +$ (higher order terms), with $(a, b) \in k^2 \setminus \{(0, 0)\}$. Set $R := k[x, y]/(f)$ and denote by $\mathfrak{m} \subset R$ the image of the ideal (x, y) in R. Show that for all prime ideals $\mathfrak{p} \subset R$ the ideal $\mathfrak{m}_{\mathfrak{p}} \subset R_{\mathfrak{p}}$ is principal. (*Hint:* First show that for any prime $\mathfrak{p} \not\subset \mathfrak{m}$ we have $\mathfrak{m}_{\mathfrak{p}} = R_{\mathfrak{p}}$.)

Exercise 3. Let $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$ and $M_n = \mathbb{Z}/(n)$, $n \geq 2$.

(i) Show that $S^{-1}M_n = 0$, for all n.
(ii) Show that the image of $(1, 1, 1, \ldots)$ under the localization map $\prod_{n\geq 2} M_n \to S^{-1}(\prod_{\geq 2} M_n)$ does not vanish.
(iii) Conclude that $S^{-1}(\prod_{\geq 2} M_n)$ and $\prod_{n\geq 2} S^{-1}M_n$ are not bijective.

[But as we saw in the lecture $S^{-1}(\bigoplus_\lambda M_\lambda) \cong \bigoplus_\lambda S^{-1}M_\lambda$.]

Exercise 4. Let R be a ring, $S \subset R$ a multiplicative subset and M an R-module.

(i) Denote by $\mathcal{S}$ the category with objects the elements of S and morphisms $\mathrm{Hom}_{\mathcal{S}}(s, t) = \{x \in R \mid xs = t\}$, for $s, t \in S$, with composition induced by multiplication. Show that $\mathcal{S}$ is a filtered category.

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(ii) Show that there is a functor $\mathcal{S} \to (R\text{-mod})$, $s \mapsto M_s =$ localization of M at s, which sends a morphism $x : s \to t$ to $e_x : M_s \to M_t$, $m/s^n \mapsto x^n m/t^n$.

(iii) Show that there are natural maps $\beta_s : M_s \to S^{-1}M$ such that $\beta_s = \beta_t \circ e_x$, for all $x : s \to t$.

(iv) Show that the β_s induce an isomorphism $\varinjlim_{s \in \mathcal{S}} M_s \xrightarrow{\simeq} S^{-1}M$.