Exercise 1. Let $R$ be a ring, $S \subset R$ a multiplicative subset and $R'$ an $R$-algebra with structure map $\varphi : R \to R'$. Set $T := \varphi(S)$.

(i) Show that $T \subset R'$ is a multiplicative subset.
(ii) Show that the ring $T^{-1}R'$ has a natural structure as $S^{-1}R'$-algebra.
(iii) By the above we can consider $T^{-1}R'$ as $S^{-1}R'$-module; we can also consider $R'$ as $R$-module and form the $S^{-1}R'$-module $S^{-1}R'$. Show that we have an isomorphism of $S^{-1}R'$-modules $T^{-1}R' \cong S^{-1}R'$.

Exercise 2. Let $k$ be a field, $k[x,y]$ the polynomial ring in two variables with coefficients in $k$ and $f \in k[x,y]$. Assume that $f = ax + by + (\text{higher order terms})$, with $(a,b) \in k^2 \setminus \{(0,0)\}$. Set $R := k[x,y]/(f)$ and denote by $m \subset R$ the image of the ideal $(x,y)$ in $R$. Show that for all prime ideals $p \subset R$ the ideal $m_p \subset R_p$ is principal. (Hint: First show that for any prime $p \not\subset m$ we have $m_p = R_p$.)

Exercise 3. Let $R = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$ and $M_n = \mathbb{Z}/(n)$, $n \geq 2$.

(i) Show that $S^{-1}M_n = 0$, for all $n$.
(ii) Show that the image of $(1, 1, 1, \ldots)$ under the localization map $\prod_{n \geq 2} M_n \to S^{-1}(\prod_{n \geq 2} M_n)$ does not vanish.
(iii) Conclude that $S^{-1}(\prod_{n \geq 2} M_n)$ and $\prod_{n \geq 2} S^{-1}M_n$ are not bijective.

[But as we saw in the lecture $S^{-1}(\bigoplus_{\lambda} M_{\lambda}) \cong \bigoplus_{\lambda} S^{-1}M_{\lambda}$]

Exercise 4. Let $R$ be a ring, $S \subset R$ a multiplicative subset and $M$ an $R$-module.

(i) Denote by $S$ the category with objects the elements of $S$ and morphisms $\text{Hom}_S(s, t) = \{x \in R \mid xs = t\}$, for $s, t \in S$, with composition induced by multiplication. Show that $S$ is a filtered category.
(ii) Show that there is a functor $\mathcal{S} \to (R\text{-mod})$, $s \mapsto M_s = \text{localization of } M \text{ at } s$, which sends a morphism $x : s \to t$ to $e_x : M_s \to M_t, m/s^n \mapsto x^n m/t^n$.

(iii) Show that there are natural maps $\beta_s : M_s \to S^{-1}M$ such that $\beta_s = \beta_t \circ e_x$, for all $x : s \to t$.

(iv) Show that the $\beta_s$ induce an isomorphism $\lim_{s \in \mathcal{S}} M_s \tilde{\to} S^{-1}M$. 