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Exercise sheet 1 for Algebra II

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Exercise 1.1 (Formal power series)**.** Let R be a ring (as usual commutative with 1). Set

$$R[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n \mid a_n \in R, n \in \mathbb{N}_0 \right\}.$$

(The sums are formal and infinite, there is no convergence condition, as a set we can identify $R[[x]]$ with the set of infinite sequences $(a_0, a_1, a_2, \ldots)$ in R.) Take $f = \sum_{n=0}^{\infty} a_n x^n$, $g = \sum_{n=0}^{\infty} b_n x^n \in R[[x]]$ and define the following operations:

$$f + g := \sum_{n=0}^{\infty} c_n x^n, \quad \text{with } c_n = a_n + b_n \in R,$$

$$f \cdot g := \sum_{n=0}^{\infty} d_n x^n, \quad \text{with } d_n = \sum_{\substack{i+j=n \\ i,j \in \mathbb{N}_0}} a_i b_j \in R.$$

(1) Show that the expression d_n above is well defined, i.e. that the sum is finite.
(2) Show that the two operations above define a ring structure on $R[[x]]$ with $0_{R[[x]]} = \sum_n 0_R \cdot x^n$ and $1_{R[[x]]} = 1_R \cdot x^0 + \sum_{n \geq 1} 0_R \cdot x^n$.
(3) There is an injective ring homomorphism $R \hookrightarrow R[[x]]$, $a \mapsto a \cdot x^0 + \sum_{n \geq 1} 0_R \cdot x^n$, and also $R \hookrightarrow R[x]$ defined by the same formula, where $R[x]$ is the polynomial ring in one variable. Show that there is a unique R-algebra homomorphism $R[x] \to R[[x]]$ which sends x to $0 \cdot x^0 + 1 \cdot x^1 + \sum_{n \geq 2} 0_R \cdot x^n$. Furthermore this map is injective.

In the following we do not write the parts of a formal sum $\sum_n a_n x^n$ which has a zero coefficient and we write x instead x^1 and $1 = x^0$.

Exercise 1.2. Let R be a ring. An element $c \in R$ is called *nilpotent* if there is a natural number $n \geq 0$ such that $c^n = 0$. We denote by $R^\times$ the group of units.

(1) Let R be a ring. Show that if $u \in R^\times$ and $c \in R$ is nilpotent, then $u + c \in R^\times$. (*Hint:* Geometric series trick!)

(2) Let $R[x]$ be the polynomial ring in one variable. Show

$$f = a_0+a_1x+\ldots+a_nx^n \in (R[x])^\times \iff a_0 \in R^\times \text{ and } a_1, \ldots, a_n \text{ are nilpotent.}$$

In particular, if R has no nilpotent elements, then $(R[x])^\times = R^\times$. (*Hint:* For $\Leftarrow$ use (i). For $\Rightarrow$: If $b_0 + \ldots + b_mx^m$ is an inverse of f show by induction on r that $a_n^{r+1}b_{m-r} = 0$, for $r \geq 0$. Deduce that a_n is nilpotent. Then use (i).)

(3) Let $R[[x]]$ be the ring of formal power series from Exercise 1.1. Show

$$f = a_0 + a_1x + a_2x^2 + \cdots \in (R[[x]])^\times \iff a_0 \in R^\times.$$

(*Hint:* First show that for $g \in R[[x]]$, the expression $1 + xg + (xg)^2 + (xg)^3 + \ldots$ is a well defined element in $R[[x]]$.)

Exercise 1.3. Which of the following ideals are equal, which are contained in another:

(1) In $\mathbb{Z}$: $< 2, 3 >$, $\mathbb{Z}$, $< 5 >$, $< 7 >$, $< 10, 15 >$

(2) In $\mathbb{Z}[x]$: $< 2, x >$, $< 2x >$, $< 9x, 4x >$, $< x^2 + x^3 >$, $< 5x^2 + x^3 >$, $< 5(x^2 + x^3) >$

(3) In $\mathbb{Q}[x]$: $< 2, x >$, $< 2x >$, $< 9x, 4x >$, $< x^2 + x^3 >$, $< 5x^2 + x^3 >$, $< 5(x^2 + x^3) >$

(4) in $\mathbb{Q}[[x]]$ (see Exercise 1.1 and 1.2, (iii)): $< 1 + x >$, $< x >$, $< \sum_{n\geq 1} x^n >$, $< 78 >$

Exercise 1.4. Let k be a field and $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ be polynomials in n variables. Set

$$X := \{a \in k^n \mid f_i(a) = 0, \text{ for all } i = 1, \ldots, r\}.$$

Denote by $\mathrm{Map}(X, k)$ the set maps from X to k. We have a map $\theta : k[x_1, \ldots, x_n] \to \mathrm{Map}(X, k)$, $f \mapsto (a \mapsto f(a))$.

(1) Show that the ring structure of k induces a ring structure on $\mathrm{Map}(X, k)$ for which θ is a ring homomorphism.

(2) Show that there is a unique ring homomorphism

$$\bar{\theta} : k[x_1, \ldots, x_n]/ < f_1, \ldots, f_r > \to \mathrm{Map}(X, k)$$

such that $\theta = \bar{\theta} \circ \pi$, where π is the quotient map $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]/ < f_1, \ldots, f_r >$.