§10 Riemann-Roch for Surfaces
Hodge-Index Thm

In this section:

k = field

X smooth projective surface $/k$
integral

$\Rightarrow$ $\omega_X := \Omega^2_{X/k} = \Lambda^2 \Omega^1_{X/k}$

is the canonical sheaf on X

K_X = can div of X, $\omega_X \cong \mathcal{O}_X(K_X)$

(1) Recall (see CSS or Ha, III, Thm. 7.11)

$P = \mathbb{P}^n_k$,

Assume $Y \overset{i}{\hookrightarrow} \mathbb{P}^n_k = P$ is a local complete intersection of codim c, i.e., locally

$\mathcal{I}_Y = \mathrm{Ker}(\mathcal{O}_P \to i_* \mathcal{O}_Y)$ is locally generated by a regular sequence of length c

Then

$$\omega_Y \cong i^*\omega_P \otimes_{\mathcal{O}_Y} \Lambda^c \left(\mathcal{I}_Y / \mathcal{I}_Y^2 \right)^\vee$$

ω_Y ← canonical sheaf of Y

$\left(\mathcal{I}_Y / \mathcal{I}_Y^2 \right)^\vee = \mathcal{H}om_{\mathcal{O}_Y}\left(\mathcal{I}_Y / \mathcal{I}_Y^2, \mathcal{O}_Y \right)$ — locally free of rk c

and $\omega_P = \Omega^n_{P/k} \cong \mathcal{O}_P(-n-1)$

② **Cor** (Adjunction formula)

X/k as above, C eff Cartier divisor on X

(which we can view as a - in general - non-irred, non-red curve on X)

$$\Rightarrow \quad \omega_C = \omega_X(C)|_C := i^*\left(\omega_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(C)\right)$$

$i : C \hookrightarrow X$

Pf: X proj $\Rightarrow$ $X \underset{i}{\hookrightarrow} P = \mathbb{P}^n_k$

X sm $\Rightarrow$

$$0 \to \mathcal{J}_X/\mathcal{J}_X^2 \to i^*\Omega^1_{P/k} \to \Omega^1_{X/k} \to 0$$

exact seq of loc free $\mathcal{O}_X$-mod (see §11, C55)

$\Rightarrow$ $\mathcal{J}_X$ loc generated by $\mathrm{rk}\,\Omega^1_{P/k} - \mathrm{rk}\,\Omega^1_{X/k} =$

$= n-2 = \mathrm{codim}(X, P)$

many elts

$\Rightarrow$ automatically reg seq (Krull Principal ideal thm + dim formulas)

$\Rightarrow$ $X \hookrightarrow P$ l.c.i $\underset{(1)}{\Rightarrow}$ $\omega_X = i^*\omega_P \otimes \Lambda^{n-2}\left(\mathcal{J}_X/\mathcal{J}_X^2\right)^\vee$ (*)

$C \hookrightarrow P$, $C \hookrightarrow X \hookrightarrow P$

$\Rightarrow$ $0 \to \mathcal{J}_X \to \mathcal{J}_C \to \mathcal{J}_C/\mathcal{J}_X \to 0$

$\mathcal{J}_C/\mathcal{J}_X$ = ideal sheaf of $C \hookrightarrow X$ = $\mathcal{O}_X(-C)$

$\mathcal{O}_X(-C)$ locally principal gen by non-zero div

$\Rightarrow$ $\mathcal{I}_C$ locally gen by a reg seq of length $n-1$

$\underset{(1)}{\Rightarrow}$ $\omega_C = \omega_P \otimes_{\mathcal{O}_P} \Lambda^{n-1}\left(\mathcal{I}_C/\mathcal{I}_C^2\right)^\vee$ (2*)

Have exact seq

$$0 \to \frac{\mathcal{I}_C^2 + \mathcal{I}_X}{\mathcal{I}_C^2} \to \mathcal{I}_C/\mathcal{I}_C^2 \longrightarrow \frac{\mathcal{I}_C}{\mathcal{I}_C^2 + \mathcal{I}_X} \to 0$$

$$\frac{\mathcal{I}_C^2 + \mathcal{I}_X}{\mathcal{I}_C^2} = \frac{\mathcal{I}_X}{\mathcal{I}_C^2 \cap \mathcal{I}_X} = \frac{\mathcal{I}_X}{\mathcal{I}_C\mathcal{I}_X + \mathcal{I}_X^2} = \left(\mathcal{I}_X/\mathcal{I}_X^2\right)|_C$$

(second equality $\Leftarrow$ use locally $\mathcal{I}_C = (s_1, \dots, s_{n-2}, \sigma)$ reg seq, $\mathcal{I}_X = (s_1 \dots s_{n-2})$)

$$\frac{\mathcal{I}_C}{\mathcal{I}_C^2 + \mathcal{I}_X} = \mathcal{O}_X(-C)|_C$$

$\underset{\Lambda^{n-1}(\,)^\vee}{\Rightarrow}$ $\Lambda^{n-1}\left(\mathcal{I}_C/\mathcal{I}_C^2\right)^\vee \cong \left(\Lambda^{n-2}\left(\mathcal{I}_X/\mathcal{I}_X^2\right)^\vee\right)|_C \otimes \mathcal{O}_X(-C)|_C$

$\underset{(*),(2*)}{\Rightarrow}$ $\omega_C = \left(\omega_X \otimes \mathcal{O}_X(-C)\right)|_C$ □

③ Cor: $C \subset X$ integral curve

$$\Rightarrow \quad K_C = (K_X + C) \cdot C \quad \text{in } CH_0(X)$$

where $\omega_C \cong \mathcal{O}_C(K_C)$

Pf: $(K_X + C) \cdot C \underset{def}{=} c_1(\omega_X(C)|_C) \underset{②}{=} c_1(\omega_C) = K_C$

④ Cor: $C \subset X$ sm, geometrically connected curve

$$\Rightarrow \quad 2g(C) - 2 = \deg((K_X + C) \cdot C)$$

Pf: $\deg K_C = 2g(C) - 2$

⑤ Example: $X = \mathbb{P}^2 \Rightarrow \omega_X = \mathcal{O}(-3)$ i.e. $K_X = -3H_0$, $H_0 = V(X_0)$

$C \subset X$ sm curve, degree $d \Rightarrow C \sim dH_1 \sim dH_0$, $H_1 = V(X_1)$

$$\Rightarrow \quad 2g(C) - 2 = \deg(d(d-3)H_0 \cdot H_1) = d(d-3)$$

$$\Rightarrow \quad g(C) = \frac{(d-1)(d-2)}{2}$$

⑥ Lem. $D \in Z^1(X)$, $k = \bar{k}$

Then $\exists$ $C_1, C_2 \subset X$ smooth irred. curves

s.t. $D \sim C_1 - C_2$

idea of proof:

1. Step: We can write $D = A_1 - A_2$
with $A_i \in Z^1(X)$ very ample (i.e. $\mathcal{O}_X(A_i)$ very ample)

indeed take L very ample
$\Rightarrow \exists n$ s.t. $\mathcal{O}_X(D) \otimes L^{\otimes n}$ gen by global sect
... $\Rightarrow (\mathcal{O}_X(D) \otimes L^{\otimes n}) \otimes L$ and $L^{\otimes n+1}$ are very ample
(see Ha, V, pf of (1.1))

2. Step A very ample on X $\Rightarrow \exists\, i : X \hookrightarrow \mathbb{P}^n_k = P$ s.t.
$\mathcal{O}_X(A) = i^* \mathcal{O}_P(H)$ H hyperplane

Bertini $\Longrightarrow \exists\, H_1 \subset P$ s.t. $X \cap H_1$ is smooth integral
(Ha, II, Thm 8.18, III, Rmk 7.19) (scheme-theoretic intersection) curve

since $\mathcal{O}_P(H) \cong \mathcal{O}_P(H_1)$

$\Rightarrow \mathcal{O}_X(A) = i^* \mathcal{O}_P(H_1) = \mathcal{O}_X(X \cap H_1)$

$\Rightarrow A \sim X \cap H_1 =: C$ $\square$

⑦ **Thm** (Riemann-Roch for surfaces)

$k = \bar{k}$, D div on X

$\Rightarrow$

$$\chi(X/k, \mathcal{O}_X(D)) - \chi(X/k, \mathcal{O}_X) = \frac{1}{2} \deg((D - K_X) \cdot D)$$

[Recall $\chi(X/k, \mathcal{O}_X(D)) = h^0(D) - h^1(D) + h^2(D)$
$\underset{S.D.}{=} h^0(D) + h^0(K_X - D) - h^1(D)$]

Pf: ⑥ $\Rightarrow$ wlog

$D = C_1 - C_2$, $C_i \subset X$ smooth integral curve

$$0 \to \underset{\mathcal{O}_X(D)}{\mathcal{O}_X(C_1 - C_2)} \longrightarrow \mathcal{O}_X(C_1) \longrightarrow \mathcal{O}_X(C_1)|_{C_2} \to 0$$

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(C_1) \to \mathcal{O}_X(C_1)|_{C_1} \to 0$$

$\Rightarrow$ §8 $\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(C_1)|_{C_1}) - \chi(\mathcal{O}_X(C_1)|_{C_2})$

RR for curves $\Rightarrow$

$$\chi(\mathcal{O}_X(C_1)|_{C_2}) = \underbrace{\deg(\mathcal{O}_X(C_1)|_{C_2})}_{\underset{\text{defn}}{=} \deg(C_1 \cdot C_2)} + 1 - g(C_2)$$

$$\chi(\mathcal{O}_X(C_1)|_{C_1}) = \deg(C_1 \cdot C_1) + 1 - g(C_1)$$

(4) $\Rightarrow$ $g(C_i) - 1 = \frac{1}{2} \deg((K_X + C_i) \cdot C_i)$

$\Rightarrow$ $\chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X)$

$$= \deg(C_1 \cdot C_1) + 1 - g(C_1) - \deg(C_1 \cdot C_2) - 1 + g(C_2)$$

$$= \deg\left(C_1 \cdot C_1 - \frac{1}{2}(K_X + C_1) \cdot C_1 - C_1 \cdot C_2 + \frac{1}{2}(K_X + C_2) \cdot C_2 \right)$$

$$= \frac{1}{2} \deg\left((C_1 - C_2 - K_X) \cdot (C_1 - C_2) \right)$$

$$= \frac{1}{2} \deg\left((D - K_X) \cdot D \right) \qquad \square$$

Lemma (8): A ample on X

$\Rightarrow \forall \; C \subset X$ integral curve

$(C . A) > 0$

proof: suff to show

$n(C . A) > 0$ for some $n \geq 1$

$\rightarrow$ wlog A very ample

i.e. define $i : X \hookrightarrow \mathbb{P}^n = P$, $\mathcal{O}_X(A) = i^* \mathcal{O}_P(H)$

$H \subset P$ hyperplane

replacing H by a different hyperplane we can assume that

$A \sim X \cap H$ intersects C properly

$\Rightarrow (C . A) = (C . (X \cap H)) \geq \#(|C| \cap |H|)$

set theoretic intersection.

and $C \cap H \neq \emptyset$

$\ulcorner$ else $C \underset{\text{closed}}{\subset} P \setminus H = \mathbb{A}^n_k \Rightarrow C$ affine $= \operatorname{Spec} A \urcorner$

C proj$/k \Rightarrow A = H^0(C, \mathcal{O}_C)$ fin. dim k-v.s.

$\llcorner \Rightarrow \dim C = \dim A = 0$ ↯ $\lrcorner$

$\square$

Lemma (9): D divisor on X, A ample on X

Assume $(D \cdot A) > (K_X \cdot A)$

$$\Rightarrow \quad H^2(X, \mathcal{O}_X(D)) = 0$$

Pf: Serre dualität $\Rightarrow$

$$H^2(X, \mathcal{O}_X(D)) \cong H^0(X, \omega_X \otimes \mathcal{O}_X(D)^\vee)^\vee$$

$$= H^0(X, \mathcal{O}(K_X - D))^\vee$$

Assume $H^2(X, \mathcal{O}_X(D)) \neq 0$

$$\Rightarrow \exists \quad 0 \neq f \in H^0(X, \mathcal{O}(K_X - D))$$

$$\Rightarrow \quad \operatorname{div}(f) + (K_X - D) \geq 0$$

if "$= 0$" $\Rightarrow (K_X - D) \cdot A = \operatorname{div}(f) \cdot A = 0$ ↯ assum.

if $> 0 \underset{(8)}{\Rightarrow} \left((\operatorname{div}(f) + (K_X - D)) \cdot A \right) > 0$

$$\Leftrightarrow \quad K_X \cdot A > D \cdot A \quad \text{↯ assump}$$

□

(10) **Cor.** $k = \bar{k}$, A ample on X

D Div on X

Assume $(D.A) > 0$, $(D.D) > 0$

$\Rightarrow \forall n \gg 0 \ \exists D' \geq 0$ s.t. $nD \sim D'$

Pf.: RR $\Rightarrow$

$$h^0(D) - h^1(D) + h^2(D) = \frac{1}{2}\left((D - K_X) \cdot D\right) + \chi(\mathcal{O}_X)$$

$(D.A) > 0 \Rightarrow \exists\, n_0$ s.t. $\forall\, n \geq n_0$:

$$(nD.A) > (K_X.A)$$

$\Rightarrow h^2(nD) = 0 \quad \forall n \geq n_0$

(9)

$\Rightarrow \forall\, n \geq n_0$:

$$h^0(nD) \geq \frac{1}{2} n^2 \underbrace{(D.D)}_{>0} - n(K_X.D) + \chi(\mathcal{O}_X)$$

$\longrightarrow \infty$ $(n \to \infty)$

$\Rightarrow h^0(nD) \neq 0 \ \forall n \gg 0 \overset{\text{by assumpt}}{\Rightarrow} \exists f: \operatorname{div}(f) + nD \geq 0$ □

(11) **Def.:** 1) D, E divisors on X

$$D \equiv E :\Longleftrightarrow D - E \equiv 0$$
$$:\Longleftrightarrow \big((D-E) \cdot F\big) = 0 \text{ in } \mathbb{Z} \quad \forall F \in CH^1(X)$$
$$(\Leftrightarrow \forall \text{ irred } F \text{ integral curve})$$

We say D and E are numerically equivalent

This defines an equivalence relation on $Z^1(X)$ (or $CH^1(X)$)

and we set

$$Num(X) := Z^1(X)/_{\equiv} = CH^1(X)/_{\equiv}$$

We obtain a non-degenerate bilinear pairing

$$Num(X) \times Num(X) \longrightarrow \mathbb{Z}$$
$$(D, E) \longmapsto (D \cdot E)$$

(12) **Thm** (Hodge Index Thm)

$k = \bar{k}$, A ample on X

Let $D \in CH^1(X)$ with $D \not\equiv 0$

Then $(D \cdot A) = 0 \implies (D \cdot D) < 0$

Pf: Else $(D \cdot D) \geq 0$.

1. case: $(D \cdot D) > 0$

$\implies \exists\, n \geq 1$ s.t. $A' := D + nA$ is ample.

(cf 1. step in proof of (6))

$\implies (A' \cdot D) = (D \cdot D) > 0 \underset{(10)}{\implies} mD \sim$ effective Div $\forall\, m \gg 0$

$\underset{(8)}{\implies} (mD \cdot A) > 0 \implies (D \cdot A) > 0$ ↯

2. case $(D . D) = 0$

$D \neq 0 \Rightarrow \exists E$ s.t. $(D . E) \neq 0$

set $E' := (A . A) E - (E . A) A$

$\Rightarrow E' . A = 0$ and $E' . D \neq 0$

(note $(A . A) > 0$ since mA reflective div, $m \geq 4$ $\underset{(8)}{\Rightarrow} m(A . A) > 0$)

set $D' := n D + E'$

$\Rightarrow (D' . A) = 0$

and $(D' . D') = \underbrace{n^2 (D . D)}_{=0} + 2n \underbrace{(D . E')}_{\neq 0} + (E' . E')$

$\Rightarrow \exists n \in \mathbb{Z}$ s.t. $(D' . D') > 0$

$\underset{\text{1. case}}{\Rightarrow}$ ↯ □

(13) Rmk: One can show
$Num(X)$ is a fin. gen group
it is torsion free (since $nD \equiv 0 \Rightarrow D \equiv 0$)

$\Rightarrow \langle\,,\rangle : Num(X) \times Num(X) \longrightarrow \mathbb{Z}$
non-deg pairing on free finite $\mathbb{Z}$-Mod.

$\otimes_{\mathbb{Z}} \mathbb{R} \Rightarrow \langle\,,\rangle : Num(X)_{\mathbb{R}} \times Num(X)_{\mathbb{R}} \longrightarrow \mathbb{R}$
non-deg quadratic form.

$\Rightarrow \exists$ basis of $Num(X)_{\mathbb{R}}$ s.t. the corresponding matrix looks like

$$\begin{pmatrix} -1 & & & & \\ & \ddots & & & \\ & & -1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

Index $(\langle,\rangle) :=$ signature$(\langle,\rangle) := (\#\{1\}, \#\{-1\})$
indep of basis (Sylvester)

Hodge Index $\Rightarrow$ matrix $\underbrace{\begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix}}_{n}$ (n

$\Rightarrow$ Index $= (1, n-1)$